augmented virtual double category

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小圏全体は pseudo double category $\mathbb{P}\mathrm{rof}$ を定めるのであった（詳しくは「pseudo double category」の PDF を参照）．即ち次のようになる．

- $\mathbb{P}\mathrm{rof}$ の対象は小圏とする．
- $\mathbb{P}\mathrm{rof}$ の垂直射は関手 $F\colon C \to D$ とする．
- $\mathbb{P}\mathrm{rof}$ の水平射は profunctor $\Phi\colon C \looparrowright D$ とする．（但し profunctor とは関手 $\Phi\colon C \to \widehat{D}$ のことである．以下これを断らずに関手 $\Phi\colon D^{\mathrm{op}} \times C \to \mathbf{Set}$ と同一視する．）
- $\mathbb{P}\mathrm{rof}$ の cell

$$\begin{tikzcd} A \arrow[r, "\Phi", "\shortmid"{marking}] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "F"] \\ B \arrow[r, "\Psi"', "\shortmid"{marking}] & D \end{tikzcd}$$

とは自然変換 $\beta\colon \Phi(-,\square) \Rightarrow \Psi(F-, G\square)$ のこととする．

これはこれでよいが，この定義では米田埋込 $y\colon C \to \widehat{C}$ が（$C = \mathbb{0}$ 以外では）垂直射にならないという問題がある．そこで（局所小とは限らない）圏全体を考える．つまり

- 対象は圏とする．
- 垂直射，水平射，cell は先と同様とする．

とすると局所小圏に対して米田埋込が定義できて，これは垂直射となるが，この定義では pseudo double category にならない．何故かというと $\Phi\colon A \looparrowright B$ と $\Psi\colon B \looparrowright C$ を profunctor としたとき，その合成 $\Psi \diamond \Phi\colon A \looparrowright C$ は左 Kan 拡張もしくはコエンドを

使って

$$\Psi \diamond \Phi(c, a) = y^\dagger \Psi \circ \Phi(c, a) \cong \int^{b \in B} \Phi(b, a) \times \Psi(c, b)$$

で定義されるが，これは B が小圏でない場合，存在するか分からないからである．

そこでこの PDF では pseudo double category の更なる一般化として augmented virtual double category というものを扱う．これを使うと米田埋込に相当する垂直射を扱うことができて，普遍随伴（定理 62, 66）が成り立つ．

※ ここで定義している言葉は参考文献の [1] とは微妙に異なる．

この PDF	[1] での用語
左 Kan 拡張	weak right Kan extension
各点左 Kan 拡張	right Kan extension
r-各点左 Kan 拡張	pointwise weak right Kan extension
r^+-各点左 Kan 拡張	pointwise right Kan extension
左 cocartesian	left cocartesian
r-cocartesian	pointwise cocartesian
左制限を持つ	has restrictions on the right
稠密な米田射	weak Yoneda morphism
各点稠密な米田射	Yoneda morphism
余連続	weakly cocontinuous
左完全	weakly left exact
r-左完全	pointwise weakly left exact
(wc)	have weakly cocartesian paths of $(0, 1)$-ary cells
(nrc)	have right nullary-cocartesian paths of $(0, 1)$-ary cells

左右が入れ替わっているのは [1] における profunctor $\Phi\colon C \to D$ の定義が（$\Phi\colon D^{\mathrm{op}} \times C \to \mathbf{Set}$ ではなく）$\Phi\colon C^{\mathrm{op}} \times D \to \mathbf{Set}$ になっているからである．

目次

1　定義

定義. **グラフ**とは 4 つ組 $G = \langle E, V, s, t\rangle$ であって以下の条件をみたすものをいう.

(1) E と V は集まりである.
(2) $s, t\colon E \to V$ は関数である.

定義. グラフ $G = \langle E, V, s, t\rangle$ における**有向パス**とは，有限列

$$\vec{J} = \langle a_0, J_1, a_1, J_2, \cdots, J_{n-1}, a_{n-1}, J_n, a_n\rangle$$

であって以下の条件をみたすものをいう.

(1) $n \geq 0$，$a_0, \cdots, a_n \in V$，$J_1, \cdots, J_n \in E$ である.
(2) $1 \leq i \leq n$ に対して $s(J_i) = a_{i-1}$，$t(J_i) = a_i$ である.

$s(\vec{J}) \coloneqq a_0$，$t(\vec{J}) \coloneqq a_n$ と書く．また n を $\vec{J}$ の長さといい $|\vec{J}|$ で表す.

定義. $\vec{J} = \langle a_0, J_1, \cdots, J_n, a_n\rangle$ と $\vec{K} = \langle a_n, K_1, \cdots, K_m, a_{n+m}\rangle$ を有向パスとするとき，$\vec{J} \frown \vec{K} \coloneqq \langle a_0, J_1, \cdots, J_n, a_n, K_1, \cdots, K_m, a_{n+m}\rangle$ と定める.

$|\vec{J}| = 0$ の場合は $\vec{J} \frown \vec{K} = \vec{K}$ である．$|\vec{K}| = 0$ の場合も同様.

グラフ G における有向パス全体の集まりを $P(G)$ で表す.

定義．augmented virtual double category $\mathbb{D}$ とは

(1) 圏 D_0 が与えられている．D_0 の対象を $\mathbb{D}$ の**対象**，D_0 の射を $\mathbb{D}$ の**垂直射**という．

(2) グラフ $G = \langle E, \mathrm{Ob}(D_0), s, t\rangle$ が与えられている．E の元を $\mathbb{D}$ の**水平射**という．また水平射 J が $s(J) = a$，$t(J) = b$ を満たすことを $J\colon a \nrightarrow b$ と描く．

(3) 集まり C と関数 $L, R\colon C \to \mathrm{Mor}(D_0)$，$s, t\colon C \to P(G)$ が与えられている．C の元を **cell** という．cell β に対して $L(\beta) = f$，$R(\beta) = g$，$s(\beta) = \vec{J}$，$t(\beta) = \vec{K}$ のときこれは次の条件を満たすものとする．

- $0 \leq |\vec{K}| \leq 1$ である．
- $a \coloneqq \mathrm{dom}(f) = s(\vec{J})$ かつ $b \coloneqq \mathrm{cod}(f) = s(\vec{K})$ である．
- $u \coloneqq \mathrm{dom}(g) = t(\vec{J})$ かつ $v \coloneqq \mathrm{cod}(g) = t(\vec{K})$ である．

またこの状況を図式では次のように描き表す．（ここで有向パスは $\longrightarrow\!\!\!\!\!\!\!\!\!+\!\!\!+\;\;\;$ で表す．）

$$\begin{tikzcd} a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, "\vec{K}"'] & v \end{tikzcd}$$

もしくは $\vec{J} = \langle a, J_1, a_1, \cdots, J_{n-1}, a_{n-1}, J_n, u\rangle$ と $\vec{K} = \langle b, K_1, \cdots, K_m, v\rangle$ を明示して

$$\begin{tikzcd} a \arrow[r, "J_1"] \arrow[d, "f"'] & a_1 \arrow[r, "J_2"] & \cdots \arrow[r, "J_{n-1}"] \arrow[d, phantom, "\Downarrow\beta"] & a_{n-1} \arrow[r, "J_n"] & u \arrow[d, "g"] \\ b \arrow[r, "K_1"'] & {} & \cdots & {} \arrow[r, "K_m"'] & v \end{tikzcd}$$

のように描く．また図式上は長さ 0 の有向パス $\langle a\rangle$ は a - - - a で表す．例えば $|\vec{K}| = 0$ の場合の cell は

$$\begin{tikzcd} a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

のように描く．

※ [1] ではこれを

$$
\begin{tikzcd}
a \arrow[rr, "\vec{J}", "|" marking] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\beta"] & u \arrow[dl, "g"] \\
& b &
\end{tikzcd}
$$

のように描いているが，この PDF では点線を使って常に四角形の図式で描く．

(4) $\mathbb{D}$ の cell

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "|" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[d, "f_1"] \\
b_0 \arrow[r, "\vec{K}_1"', "|" marking] & b_1
\end{tikzcd}
\quad , \quad \cdots \quad , \quad
\begin{tikzcd}
a_{n-1} \arrow[r, "\vec{J}_n", "|" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_{n-1} \arrow[r, "\vec{K}_n"', "|" marking] & b_n
\end{tikzcd}
$$

と

$$
\begin{tikzcd}
b_0 \arrow[r, "\vec{H}", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\gamma"] & b_n \arrow[d, "h"] \\
c \arrow[r, "\vec{L}"', "|" marking] & d
\end{tikzcd}
$$

が $\vec{K}_1 \frown \cdots \frown \vec{K}_n = \vec{H}$ を満たすとき，これらの合成と呼ばれる cell

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1 \frown \cdots \frown \vec{J}_n", "|" marking] \arrow[d, "g \circ f_0"'] \arrow[dr, phantom, "\Downarrow\frac{\beta_1 \cdots \beta_n}{\gamma}"] & a_n \arrow[d, "h \circ f_n"] \\
c \arrow[r, "\vec{L}"', "|" marking] & d
\end{tikzcd}
$$

が与えられている．この cell を図式では

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "|" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "|" marking] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "|" marking] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n", "|" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{K}_1"', "|" marking] \arrow[d, "g"'] & b_1 \arrow[r, "\vec{K}_2"', "|" marking] & \cdots \arrow[r, "\vec{K}_{n-1}"', "|" marking] \arrow[d, phantom, "\Downarrow\gamma"] & b_{n-1} \arrow[r, "\vec{K}_n"', "|" marking] & b_n \arrow[d, "h"] \\
c \arrow[rrrr, "\vec{L}"', "|" marking] & & & & d
\end{tikzcd}
$$

とも描く．

(5) 合成できる cell に対して結合律

$$\frac{\dfrac{\beta_{11}\cdots\beta_{1n_1}}{\gamma_1}\cdots\dfrac{\beta_{m1}\cdots\beta_{mn_m}}{\gamma_m}}{\sigma} = \frac{\beta_{11}\cdots\beta_{mn_m}}{\dfrac{\gamma_1\cdots\gamma_m}{\sigma}}$$

が成り立つ.

(6) 任意の水平射 $J\colon a \nrightarrow b$ に対して，cell

$$\begin{tikzcd}
a \arrow[r, "J", "|" marking] \arrow[d, "\mathrm{id}_a"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & b \arrow[d, "\mathrm{id}_b"] \\
a \arrow[r, "J"', "|" marking] & b
\end{tikzcd}$$

が与えられている.

(7) 任意の垂直射 $f\colon a \to b$ に対して，cell

$$\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}$$

が与えられている.

(8) 次の図式のような cell β に対して，次の等式が成り立つ.

$$\begin{tikzcd}
a \arrow[r, "\vec{J}", "||" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, dashed, no head] \arrow[d, "\mathrm{id}_b"'] \arrow[dr, phantom, "\Downarrow U\mathrm{id}_b"] & b \arrow[d, "\mathrm{id}_b"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "\vec{J}", "||" marking] \arrow[dd, "f"'] \arrow[ddr, phantom, "\Downarrow\beta"] & u \arrow[dd, "g"] \\
 & \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}$$

(9) 次の図式のような cell β に対して，次の等式が成り立つ.

$$\begin{tikzcd}
a \arrow[r, "\vec{J}", "||" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K", "|" marking] \arrow[d, "\mathrm{id}_b"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_K"] & v \arrow[d, "\mathrm{id}_v"] \\
b \arrow[r, "K"', "|" marking] & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "\vec{J}", "||" marking] \arrow[dd, "f"'] \arrow[ddr, phantom, "\Downarrow\beta"] & u \arrow[dd, "g"] \\
 & \\
b \arrow[r, "K"', "|" marking] & v
\end{tikzcd}$$

(10) 次の図式のような cell β に対して，次の等式が成り立つ．

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "\mathrm{id}_a"'] \arrow[dr, phantom, "\Downarrow U\mathrm{id}_a"] & a \arrow[d, "\mathrm{id}_a"] \\
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "g"] \\
b \arrow[r, "\vec{K}"', "\shortmid\shortmid" marking] & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[dd, "f"'] \arrow[dd, phantom, shift left=6, "\Downarrow\beta"] & a \arrow[dd, "g"] \\
& \\
b \arrow[r, "\vec{K}"', "\shortmid\shortmid" marking] & v
\end{tikzcd}
$$

(11) 次の図式のような cell β $(n \geq 1)$ に対して，次の等式が成り立つ．

$$
\begin{tikzcd}
a_0 \arrow[r, "J_1", "\shortmid" marking] \arrow[d, "\mathrm{id}_{a_0}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{J_1}"] & a_1 \arrow[r, "J_2", "\shortmid" marking] \arrow[d, "\mathrm{id}_{a_1}"] & \cdots \arrow[r, "J_{n-1}", "\shortmid" marking] & a_{n-1} \arrow[r, "J_n", "\shortmid" marking] \arrow[d, "\mathrm{id}_{a_{n-1}}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{J_n}"] & a_n \arrow[d, "\mathrm{id}_{a_n}"] \\
a_0 \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "g"'] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\beta"] & a_{n-1} \arrow[r, "J_n"', "\shortmid" marking] & a_n \arrow[d, "h"] \\
b \arrow[rrrr, "\vec{K}"', "\shortmid\shortmid" marking] & & & & v
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "J_1", "\shortmid" marking] \arrow[dd, "g"'] & \cdots \arrow[r, "J_n", "\shortmid" marking] \arrow[dd, phantom, "\Downarrow\beta"] & a_n \arrow[dd, "h"] \\
& & \\
b \arrow[rr, "\vec{K}"', "\shortmid\shortmid" marking] & & v
\end{tikzcd}
$$

(12) 次の図式のような cell $\beta_1, \cdots, \beta_n$ と γ に対して，次の等式が成り立つ．

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortmid\shortmid" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "\shortmid\shortmid" marking] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{i-1}", "\shortmid\shortmid" marking] & a_{i-1} \arrow[r, "\vec{J}_i", "\shortmid\shortmid" marking] \arrow[d, "f_{i-1}"'] \arrow[dr, phantom, "\Downarrow\beta_i"] & a_i \arrow[r, dashed, no head] \arrow[d, "f_i"] \arrow[dr, phantom, "\Downarrow Uf_i"] & a_i \arrow[r, "\vec{J}_{i+1}", "\shortmid\shortmid" marking] \arrow[d, "f_i"] \arrow[dr, phantom, "\Downarrow\beta_{i+1}"] & a_{i+1} \arrow[r, "\vec{J}_{i+2}", "\shortmid\shortmid" marking] \arrow[d, "f_{i+1}"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortmid\shortmid" marking] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid\shortmid" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{K}_1"', "\shortmid\shortmid" marking] \arrow[d, "g"'] & b_1 \arrow[r, "\vec{K}_2"', "\shortmid\shortmid" marking] & \cdots \arrow[r, "\vec{K}_{i-1}"', "\shortmid\shortmid" marking] & b_{i-1} \arrow[r, "\vec{K}_i"', "\shortmid\shortmid" marking] & b_i \arrow[r, dashed, no head] \arrow[d, phantom, "\Downarrow\gamma"] & b_i \arrow[r, "\vec{K}_{i+1}"', "\shortmid\shortmid" marking] & a_{i+1} \arrow[r, "\vec{K}_{i+2}"', "\shortmid\shortmid" marking] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortmid\shortmid" marking] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid\shortmid" marking] & b_n \arrow[d, "h"] \\
c \arrow[rrrrrrrrr, "\vec{L}"', "\shortmid\shortmid" marking] & & & & & & & & & v
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortmid\shortmid" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "\shortmid\shortmid" marking] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortmid\shortmid" marking] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid\shortmid" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{K}_1"', "\shortmid\shortmid" marking] \arrow[d, "g"'] & b_1 \arrow[r, "\vec{K}_2"', "\shortmid\shortmid" marking] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortmid\shortmid" marking] \arrow[d, phantom, "\Downarrow\gamma"] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid\shortmid" marking] & b_n \arrow[d, "h"] \\
c \arrow[rrrr, "\vec{L}"', "\shortmid\shortmid" marking] & & & & v
\end{tikzcd}
$$

(13) 垂直射 $a \xrightarrow{f} b \xrightarrow{g} c$ に対して

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, dashed, no head] \arrow[d, "f"'] & & a \arrow[d, "f"] \\
b \arrow[d, "g"'] \arrow[rr, phantom, "\Downarrow U(g\circ f)"] & & b \arrow[d, "g"] \\
c \arrow[rr, dashed, no head] & & c
\end{tikzcd}
$$

が成り立つ.

定義. augmented virtual double category の定義において，cell

$$
\begin{tikzcd}
a \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "\vec{K}"', "\shortmid"{marking}] & v
\end{tikzcd}
$$

においては必ず $|\vec{K}| = 1$ であるとして，更に Uf に関する条件を取り除いたものを **virtual double category**（もしくは fc-multicategory[*1]）という.

例 1. virtual double category では水平射が合成できる必要はないため

- $\mathbb{D}$ の対象は（局所小とは限らない）圏とする.
- $\mathbb{D}$ の垂直射は関手とする.
- $\mathbb{D}$ の水平射は profunctor とする.

により virtual double category $\mathbb{D}$ を定めることができる．より詳しくは以下のようにする.

- $n \geq 1$ の場合，$\mathbb{D}$ の cell

$$
\begin{tikzcd}
C_0 \arrow[r, "\Phi_1", "\shortmid"{marking}] \arrow[d, "G"'] & C_1 \arrow[r, "\Phi_2", "\shortmid"{marking}] & \cdots \arrow[r, "\Phi_{n-1}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\beta"] & C_{n-1} \arrow[r, "\Phi_n", "\shortmid"{marking}] & C_n \arrow[d, "F"] \\
D_0 \arrow[rrrr, "\Psi"', "\shortmid"{marking}] & & & & D_1
\end{tikzcd}
$$

[*1] 一般にある種のモナド T に対して T-multicategory というものを定義することができ，T として自由圏を取るモナド $\mathrm{fc}\colon \mathbf{Graph} \to \mathbf{Graph}$ を考えた fc-multicategory が virtual double category になる.

とは，写像の族 $\beta = \{\beta_{\vec{a}}\}_{\vec{a}}$ であって，各 $a_i \in C_i$ について自然[*2]なもののこととする．但し添え字の $\vec{a}$ は組 $\langle a_0, \cdots, a_n \rangle$ を表し，また $\beta_{\vec{a}}$ は

$$\beta_{\vec{a}} \colon \Phi_1(a_1, a_0) \times \cdots \times \Phi_n(a_n, a_{n-1}) \to \Psi(Fa_n, Ga_0)$$

という写像である．

- $\mathbb{D}$ の cell

$$\begin{tikzcd}
C \arrow[r, dashed, no head] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "F"] \\
D_0 \arrow[r, "\Psi"', "|" marking] & D_1
\end{tikzcd}$$

とは，写像の族 $\beta = \{\beta_a \colon 1 \to \Psi(Fa, Ga)\}_a$ であって，$a \in C$ について自然なもののこととする．

- $\beta^1, \cdots, \beta^n$ と γ を

$$\begin{tikzcd}
A^i_0 \arrow[r, "\Phi^i_1", "|" marking] \arrow[d, "F^{i-1}"'] & A^i_1 \arrow[r, "\Phi^i_2", "|" marking] & \cdots \arrow[r, "\Phi^i_{n^i-1}", "|" marking] \arrow[d, phantom, "\Downarrow\beta^i"] & A^i_{n^i-1} \arrow[r, "\Phi^i_{n^i}", "|" marking] & A^i_{n^i} \arrow[d, "F^i"] \\
B^{i-1} \arrow[rrrr, "\Psi^i"', "|" marking] & & & & B^i
\end{tikzcd}$$

$$\begin{tikzcd}
B^0 \arrow[r, "\Psi^1", "|" marking] \arrow[d, "H"'] & B^1 \arrow[r, "\Psi^2", "|" marking] & \cdots \arrow[r, "\Psi^{n-1}", "|" marking] \arrow[d, phantom, "\Downarrow\gamma"] & B^{n-1} \arrow[r, "\Psi^n", "|" marking] & B^n \arrow[d, "G"] \\
C \arrow[rrrr, "\Sigma"', "|" marking] & & & & D
\end{tikzcd}$$

という cell とする．$a^i_j \in A^i_j$ に対して

$$\vec{a}^1 \coloneqq \langle a^1_0, \cdots, a^1_{n^1-1}, a^2_0 \rangle, \quad \vec{a}^2 \coloneqq \langle a^2_0, \cdots, a^2_{n^2-1}, a^3_0 \rangle, \quad \cdots,$$
$$\vec{a}^{n-1} \coloneqq \langle a^{n-1}_0, \cdots, a^{n-1}_{n^{n-1}-1}, a^n_0 \rangle, \quad \vec{a}^n \coloneqq \langle a^n_0, \cdots, a^n_{n^n-1}, a^n_{n^n} \rangle$$

[*2] 例えば a_1 について自然というのは cowedge になっているということである．「エンド」や「豊穣圏」の PDF を参照．

と書いたとき，写像 $\left(\dfrac{\beta^1\cdots\beta^n}{\gamma}\right)_{a_0^1\cdots a_{n^1-1}^1 a_0^2 a_1^2\cdots a_{n^n-1}^n a_{n^n}^n}$ を合成

$$\begin{aligned}
&[\Phi_1^1(a_1^1, a_0^1)\times\cdots\times\Phi_{n^1}^1(a_0^2, a_{n^1-1}^1)]\times[\Phi_1^2(a_1^2, a_0^2)\times\cdots\\
&\quad\times\Phi_1^{n-1}(a_0^n, a_{n^{n-1}-1}^{n-1})]\times[\Phi_1^n(a_1^n, a_0^n)\times\cdots\times\Phi_{n^n}^n(a_{n^n}^n, a_{n^n-1}^n)]\\
&\quad\xrightarrow{\beta_{\vec{a}^1}^1\times\cdots\times\beta_{\vec{a}^n}^n}\Psi^1(F^1a_0^2, F^0a_0^1)\times\cdots\times\Psi^n(F^na_{n^n}^n, F^{n-1}a_0^n)\\
&\quad\xrightarrow{\gamma_{F^0a_0^1,\cdots,F^{n-1}a_0^n,F^na_{n^n}^n}}\Psi(GF^na_{n^n}^n, HF^0a_0^1)
\end{aligned}$$

で定める．これは $a_j^i\in A_j^i$ について自然だから cell

$$\begin{tikzcd}
A_0^1 \arrow[r, "\vec{\Phi}^1\frown\cdots\frown\vec{\Phi}^n", "|"'] \arrow[d, "HF^n"'] \arrow[dr, phantom, "\Downarrow\frac{\beta^1\cdots\beta^n}{\gamma}"] & A_{n^n}^n \arrow[d, "GF^0"] \\
C \arrow[r, "\Sigma"', "|"] & D
\end{tikzcd}$$

を定める．

- 写像の合成は結合律を満たすから，条件 (5) が成り立つ．
- profunctor $\Phi\colon C\to D$ に対して恒等変換 id_Φ は明らかに cell

$$\begin{tikzcd}
C \arrow[r, "\Phi", "|"'] \arrow[d, "\mathrm{id}_C"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_\Phi"] & D \arrow[d, "\mathrm{id}_D"] \\
C \arrow[r, "\Phi"', "|"] & D
\end{tikzcd}$$

を定める．

- この id_Φ は明らかに条件 (9), (11) を満たす．

ここで実は同様にして augmented virtual double category $\mathbb{P}$ROF も定めることができる．即ち次のようにする．

- $\mathbb{P}$ROF の cell

$$\begin{tikzcd}
C_0 \arrow[r, "\Phi_1", "|"'] \arrow[d, "G"'] & C_1 \arrow[r, "\Phi_2", "|"'] & \cdots \arrow[r, "\Phi_{n-1}", "|"'] \arrow[d, phantom, "\Downarrow\beta"] & C_{n-1} \arrow[r, "\Phi_n", "|"'] & C_n \arrow[d, "F"] \\
D \arrow[rrrr, dashed, no head] & & & & D
\end{tikzcd}$$

とは，上記の cell の定義において Ψ として $\mathrm{Hom}_D(-,\square)$ を取ったものとする．（但し D が局所小圏でない場合，$\mathrm{Hom}_D(Fa_n, Ga_0)$ は集合にならない可能性があることに注意.）

- $\mathbb{P}$ROF の合成は次のようにする．例えば β^1, β^2, γ が

$$
\begin{tikzcd}
A_0 \arrow[r, "\Phi^1", "|"{marking}] \arrow[d, "F^0"'] \arrow[dr, phantom, "\Downarrow\beta^1"] & A_1 \arrow[d, "F^1"] \\
B_0 \arrow[r, "\Psi"', "|"{marking}] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\Phi^2", "|"{marking}] \arrow[d, "F^1"'] \arrow[dr, phantom, "\Downarrow\beta^2"] & A_2 \arrow[d, "F^2"] \\
B_1 \arrow[r, dashed, no head] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0 \arrow[r, "\Psi", "|"{marking}] \arrow[d, "H"'] \arrow[dr, phantom, "\Downarrow\gamma"] & B_1 \arrow[d, "G"] \\
C_0 \arrow[r, "\Sigma"', "|"{marking}] & C_1
\end{tikzcd}
$$

の場合，$\gamma'_{b_0b_1b_2} \colon \Psi(b_1,b_0) \times \mathrm{Hom}_{B_1}(b_2,b_1) \to \Sigma(Gb_2, Hb_0)$ を

$$\gamma'_{b_0b_1b_2}(x,f) \coloneqq \gamma_{b_0b_1}(\Psi(f,\mathrm{id})(x))$$

により定める．この $\gamma'_{b_0b_1b_2}$ は b_0, b_1, b_2 について自然である．そこで合成

$$
\begin{aligned}
&\Phi^1(a_1,a_0) \times \Phi^2(a_2,a_1) \\
&\qquad \xrightarrow{\beta^1_{a_0a_1} \times \beta^2_{a_1a_2}} \Psi(F^1a_1, F^0a_0) \times \mathrm{Hom}_{B_1}(F^2a_2, F^1a_1) \\
&\qquad \xrightarrow{\gamma'_{F^0a_0, F^1a_1, F^2a_2}} \Sigma(GF^2a_2, HF^0a_0)
\end{aligned}
$$

を考えればこれが cell $\frac{\beta^1\beta^2}{\gamma}$ を定める．また例えば

$$
\begin{tikzcd}
A_0 \arrow[r, "\Phi^1", "|"{marking}] \arrow[d, "F^0"'] \arrow[dr, phantom, "\Downarrow\beta^1"] & A_1 \arrow[d, "F^1"] \\
B \arrow[r, dashed, no head] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\Phi^2", "|"{marking}] \arrow[d, "F^1"'] \arrow[dr, phantom, "\Downarrow\beta^2"] & A_2 \arrow[d, "F^2"] \\
B \arrow[r, dashed, no head] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, dashed, no head] \arrow[d, "H"'] \arrow[dr, phantom, "\Downarrow\gamma"] & B \arrow[d, "G"] \\
C_0 \arrow[r, "\Sigma"', "|"{marking}] & C_1
\end{tikzcd}
$$

の場合は $\gamma_b \colon 1 \to \Sigma(Gb, Hb)$ となるから

$$\gamma''_{b_0b_1b_2} \colon \mathrm{Hom}_B(b_1,b_0) \times \mathrm{Hom}_B(b_2,b_1) \to \Sigma(Gb_2, Hb_0)$$

を

$$\gamma''_{b_0b_1b_2}(g,f) \coloneqq \Sigma(G(g\circ f), \mathrm{id}_{Hb_0})(\gamma_{b_0}(*))$$

（$b_2 \xrightarrow{f} b_1 \xrightarrow{g} b_0$ だから $\Sigma(G(g\circ f), \mathrm{id}_{Hb_0}) \colon \Sigma(Gb_0, Hb_0) \to \Sigma(Gb_2, Hb_0)$ となり $\gamma''_{b_0b_1b_2}(g,f) \in \Sigma(Gb_2, Hb_0)$ である.）

により定める．この $\gamma'_{b_0b_1b_2}$ は b_0, b_1, b_2 について自然である．そこで合成

$$\begin{aligned}
&\Phi^1(a_1, a_0) \times \Phi^2(a_2, a_1)\\
&\qquad \xrightarrow{\beta^1_{a_0a_1} \times \beta^2_{a_1a_2}} \mathrm{Hom}_B(F^1a_1, F^0a_0) \times \mathrm{Hom}_B(F^2a_2, F^1a_1)\\
&\qquad \xrightarrow{\gamma''_{F^0a_0, F^1a_1, F^2a_2}} \Sigma(GF^2a_2, HF^0a_0)
\end{aligned}$$

を考えればこれも cell $\frac{\beta^1\beta^2}{\gamma}$ を定める．より一般の場合の合成も同様に定義する．

- これはやはり条件 (5) を満たす．
- 関手 $F\colon C \to D$ に対して恒等自然変換 id_F は，$(\mathrm{id}_F)_c\colon 1 \to \mathrm{Hom}_D(Fc, Fc)$ とみなせば cell

$$\begin{tikzcd}
C \arrow[r, dashed, no head] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_F"] & C \arrow[d, "F"] \\
D \arrow[r, "\Sigma"', "\shortmid" marking] & D
\end{tikzcd}$$

を定める．これを UF とする．

- この UF は (8), (10), (12), (13) を満たす．

このようにする利点は，cell として全ての自然変換が含まれることにある．つまり cell として

$$\begin{tikzcd}
C \arrow[r, dashed, no head] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow \beta"] & C \arrow[d, "F"] \\
D \arrow[r, dashed, no head] & D
\end{tikzcd}$$

を考えれば，これは自然変換 $\beta\colon F \Rightarrow G$ のことである．つまり $\mathbb{P}\mathrm{ROF}$ は圏のなす strict 2-category $\mathbf{CAT}$ の情報を完全に含んでいる．より一般に augmented virtual double category $\mathbb{D}$ に対して vertical 2-category $\mathcal{V}(\mathbb{D})$ を定義することができる（後述）．

virtual double category の場合は，もし各対象 a に対して「単位元」となる水平射 $I_a\colon a \nrightarrow a$ が存在すれば vertical 2-category を定義することができる．例えば profunctor の場合は「単位元」は $\mathrm{Hom}_C(-, \square)$ であるが，C が局所小でなければこれは profunctor にならない．（もし C が局所小であれば cell

$$\begin{tikzcd}
C \arrow[r, dashed, no head] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow \beta"] & C \arrow[d, "F"] \\
D \arrow[r, "\mathrm{Hom}_D"', "\shortmid" marking] & D
\end{tikzcd}$$

が自然変換 $\beta\colon F \Rightarrow G$ となる．）従ってこの virtual double category は **CAT** の情報を完全には含んでいない．

この問題を解決する方法として，profunctor $\Phi\colon D^{\mathrm{op}} \times C \to \mathbf{Set}$ の行き先は集合でなくてもよい，とする方法がある．そのようにしてもやはり virtual double category にはなるが，これだと今度は水平射が「多すぎる」という問題が発生する*3．そこで admissible という条件を導入したのが米田構造である． □

例 2. モノイダル圏 V*4に対して V-profunctor がなす augmented virtual double category を定義したい．そこで V-$\mathbb{P}$ROF を次のように定義する．

- V-$\mathbb{P}$ROF の対象は V-豊穣圏とする．
- V-$\mathbb{P}$ROF の垂直射は V-関手とする．
- V-$\mathbb{P}$ROF の水平射は V-関手 $\Phi\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ とする．
- $n \geq 1$ の場合，V-$\mathbb{P}$ROF の cell

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\Phi_1"] \arrow[d, "G"'] & \mathcal{C}_1 \arrow[r, "\Phi_2"] & \cdots \arrow[r, "\Phi_{n-1}"] \arrow[d, phantom, "\Downarrow\beta"] & \mathcal{C}_{n-1} \arrow[r, "\Phi_n"] & \mathcal{C}_n \arrow[d, "F"] \\
\mathcal{D} \arrow[rrrr, "\Psi"'] & & & & \mathcal{E}
\end{tikzcd}$$

とは，各 $a_i \in \mathcal{C}_i$ について自然な V の射の族

$$\beta = \{\beta_{a_0 \cdots a_n}\colon \Phi_1(a_1, a_0) \otimes \cdots \otimes \Phi_n(a_n, a_{n-1}) \to \Psi(Fa_n, Ga_0)\}_{a_0 \cdots a_n}$$

のこととする．

- V-$\mathbb{P}$ROF の cell

$$\begin{tikzcd}
\mathcal{C} \arrow[r, dashed, no head] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\beta"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D}_0 \arrow[r, "\Psi"'] & \mathcal{D}_1
\end{tikzcd}$$

とは $a \in \mathcal{C}$ について自然な V の射の族

$$\beta = \{\beta_a\colon I \to \Psi(Fa, Ga)\}_a$$

のこととする．

---

*3 これは例 54 で分かる．

*4 ここでいうモノイダル圏は「豊穣圏」の PDF と同様，完備かつ余完備な対称モノイダル閉圏である．

- 合成は例 1 と同様に定める．

これにより V-$\mathbb{P}$ROF は virtual double category となる．この定義では任意の V-豊穣圏 $\mathcal{C}$ に対して V-profunctor $\mathcal{C}(-,\square)$ が「単位元」となるから，例 1 と同様に V-$\mathbb{P}$ROF は V-**CAT** の情報を含んでいる．但しこの定義では小 V-豊穣圏にしか米田埋込を（垂直射として）考えることはできない．そこでモノイダル圏 V' を次の条件を満たすように取る[*5]．

(1) $V \subset V'$ は部分圏であり，包含関手 $V \to V'$ は対称 strong モノイダル関手である．更にこれは任意の極限と交換する．
(2) V における（小とは限らない）任意の極限が V' の中に存在する．
(3) $u \in V$，$v \in V'$ で $u \cong v$ となるならば $v \in V$ である．

この V' に対して virtual double category V'-$\mathbb{P}$ROF を考えれば，任意の V-豊穣圏 $\mathcal{C}$ に対して米田埋込 $y\colon \mathcal{C} \to \widehat{\mathcal{C}} \coloneqq [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ が垂直射になる．勿論これも「単位元」$\mathcal{C}(-,\square)$ が水平射になるから，V'-$\mathbb{P}$ROF は V'-**CAT** を含んでいる．

ところがこの定義では水平射が「多すぎる」[*6]．これは水平射が V'-profunctor，つまり V'-関手 $\Phi\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}'$ になるからである．そこで水平射は V'-関手 $\Phi\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ のみを考えたい．しかしそうすると今度は「単位元」$\mathcal{C}(-,\square)$ が V-豊穣圏に対してしか考えられなくなってしまう．そこで登場するのが augmented virtual double category (V, V')-$\mathbb{P}$ROF である．これは次のように定義する．

- (V, V')-$\mathbb{P}$ROF の対象は V'-豊穣圏とする．
- (V, V')-$\mathbb{P}$ROF の垂直射は V'-関手とする．
- (V, V')-$\mathbb{P}$ROF の水平射は V'-関手 $\Phi\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$（これも V-profunctor と呼ぶ）とする．
- $n \geq 1$ の場合，(V, V')-$\mathbb{P}$ROF の cell

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\Phi_1", "|" marking] \arrow[d, "G"'] & \mathcal{C}_1 \arrow[r, "\Phi_2", "|" marking] & \cdots \arrow[r, "\Phi_{n-1}", "|" marking] \arrow[d, phantom, "\Downarrow\beta"] & \mathcal{C}_{n-1} \arrow[r, "\Phi_n", "|" marking] & \mathcal{C}_n \arrow[d, "F"] \\
\mathcal{D} \arrow[rrrr, "\Psi"', "|" marking] & & & & \mathcal{E}
\end{tikzcd}
$$

[*5] このような V' を考えることはできるのか怪しいと思う人もいるかもしれない．これは厳密には Grothendieck 宇宙を取って考える必要がある．ここでは素朴にこのような V' が取れると思っておく．詳しくは「Universe Enlargement」の PDF を参照．
[*6] これも例 54 を参照.

とは，各 $a_i \in \mathcal{C}_i$ について自然な V の射の族

$$\beta = \{\beta_{a_0\cdots a_n} \colon \Phi_1(a_1, a_0) \otimes \cdots \otimes \Phi_n(a_n, a_{n-1}) \to \Psi(Fa_n, Ga_0)\}_{a_0\cdots a_n}$$

のこととする．

- (V, V')-$\mathbb{P}$ROF の cell

$$\begin{tikzcd}
\mathcal{C} \arrow[r, dashed, no head] \arrow[d, "G"'] \arrow[r, phantom, shift right=6ex, "\Downarrow\beta"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D}_0 \arrow[r, "\Psi"', "\shortmid"{marking}] & \mathcal{D}_1
\end{tikzcd}$$

とは $a \in \mathcal{C}$ について自然な V の射の族

$$\beta = \{\beta_a \colon I \to \Psi(Fa, Ga)\}_a$$

のこととする．

- (V, V')-$\mathbb{P}$ROF の cell

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\Phi_1", "\shortmid"{marking}] \arrow[d, "G"'] & \mathcal{C}_1 \arrow[r, "\Phi_2", "\shortmid"{marking}] & \cdots \arrow[r, "\Phi_{n-1}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\beta"] & \mathcal{C}_{n-1} \arrow[r, "\Phi_n", "\shortmid"{marking}] & \mathcal{C}_n \arrow[d, "F"] \\
\mathcal{D} \arrow[rrrr, dashed, no head] & & & & \mathcal{D}
\end{tikzcd}$$

とは，上記の Ψ として $\mathcal{D}(F-, G\Box)$ を使ったものとする．（但しこの場合，β は V' の射の族を考える．）

このようにすれば (V, V')-$\mathbb{P}$ROF は V'-**CAT** を含みながら水平射が「多すぎない」ようにすることができる． $\Box$

augmented virtual double category $\mathbb{D}$ においては 2 つの cell

$$\begin{tikzcd}
a \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "\vec{K}"', "\shortmid\shortmid"{marking}] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "\vec{J'}", "\shortmid\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
v \arrow[r, "\vec{K}'"', "\shortmid\shortmid"{marking}] & t
\end{tikzcd}$$

があったとしても，その水平合成

$$\begin{tikzcd}
a \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "\vec{J'}", "\shortmid\shortmid"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
b \arrow[r, "\vec{K}"', "\shortmid\shortmid"{marking}] & v \arrow[r, "\vec{K}'"', "\shortmid\shortmid"{marking}] & t
\end{tikzcd}$$

は考えることができない．$|\vec{K} \frown \vec{K}'| \leq 1$ とは限らないからである．但し $|\vec{K}| = |\vec{K}'| = 0$ の場合，即ち

$$
\begin{tikzcd}
a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "\vec{J}'"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

の場合は，$\vec{K} \frown \vec{K}' = \vec{K} = \vec{K}'$ なので，合成

$$
\begin{tikzcd}
a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "\vec{J}'"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
b \arrow[r, dash, dashed] \arrow[d, equal] & b \arrow[r, dash, dashed] \arrow[d, phantom, "\Downarrow U(\mathrm{id}_b)"] & b \arrow[d, equal] \\
b \arrow[rr, dash, dashed] & & b
\end{tikzcd}
$$

を考えることができる（ここで垂直射 id_b を等号で表した）．この cell を以下では単に

$$
\begin{tikzcd}
a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "\vec{J}'"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

で表す．$|\vec{K}| + |\vec{K}'| = 1$ の場合も同様の記法を使う．例えば $|\vec{K}| = 1$ で $|\vec{K}'| = 0$ の場合は

$$
\begin{tikzcd}
a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "\vec{J}'"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
b \arrow[r, "K"'] & v \arrow[r, dash, dashed] & v
\end{tikzcd}
:=
\begin{tikzcd}
a \arrow[r, "\vec{J}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "\vec{J}'"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta'"] & s \arrow[d, "h"] \\
b \arrow[r, "K"'] \arrow[d, equal] & v \arrow[r, dash, dashed] \arrow[d, phantom, "\Downarrow\mathrm{id}_K"] & v \arrow[d, equal] \\
b \arrow[rr, "K"'] & & v
\end{tikzcd}
$$

である．

このことを使うと augmented virtual double category $\mathbb{D}$ に対して，次のように strict 2-category $\mathcal{V}(\mathbb{D})$ を定義することができる．（この $\mathcal{V}(\mathbb{D})$ を **vertical 2-category** という．）

- $\mathcal{V}(\mathbb{D})$ の対象は $\mathbb{D}$ の対象とする．
- $\mathcal{V}(\mathbb{D})$ の 1-morphism は $\mathbb{D}$ の垂直射とする．

- $\mathcal{V}(\mathbb{D})$ の 2-morphism $\beta\colon f \Rightarrow g$ は次の形の cell とする．

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

- 2-morphism の垂直合成は上で説明した合成とする．即ち $\beta\colon f \Rightarrow g$，$\gamma\colon g \Rightarrow h$ に対して

$$
\gamma * \beta \quad := \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
$$

である．

- $a, b \in \mathbb{D}$ に対して $\mathcal{V}(\mathbb{D})(a,b)$ は圏である．

$\because$) まず結合律を示す．そのために $f, g, h, k\colon a \to b$ として $\beta\colon f \Rightarrow g$，$\gamma\colon g \Rightarrow h$，$\sigma\colon h \Rightarrow k$ とする．このとき条件 (12) を使えば

$$
(\sigma * \gamma) * \beta \quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, dashed, no head] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[rr, dashed, no head] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow U\mathrm{id}"] & & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\
b \arrow[rrr, dashed, no head] & & & b
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, dashed, no head] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\
b \arrow[rrr, dashed, no head] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow U\mathrm{id}"] & & & b \arrow[d, equal] \\
b \arrow[rrr, dashed, no head] & & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, dashed, no head] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[rr, dashed, no head] & & b \arrow[d, equal] \\
b \arrow[rrr, dashed, no head] & & & b
\end{tikzcd}
$$

$$= \quad \sigma * (\gamma * \beta)$$

である．恒等射についても (12) より

$$Ug * \beta \quad = \quad \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & a \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow \beta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\ b \arrow[rr, dashed, no head] & & b \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \beta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] \\ b \arrow[r, dashed, no head] & b \end{tikzcd} \quad = \quad \beta$$

$$\beta * Uf \quad = \quad \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \beta"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\ b \arrow[rr, dashed, no head] & & b \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \beta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] \\ b \arrow[r, dashed, no head] & b \end{tikzcd} \quad = \quad \beta$$

となり Uf が f の恒等射となる．

- 1-morphism の合成は垂直射の合成として，2-morphism の水平合成は cell の合成とする．これは関手 $M^{abc}\colon \mathcal{V}(\mathbb{D})(b,c) \times \mathcal{V}(\mathbb{D})(a,b) \to \mathcal{V}(\mathbb{D})(a,c)$ を定める．

∵) $\mathcal{V}(\mathbb{D})$ において次の状況のとき $(\tau * \sigma) \bullet (\gamma * \beta) = (\tau \bullet \gamma) * (\sigma \bullet \beta)$ を示す．

$$\begin{tikzcd}[column sep=large] a \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\beta\Downarrow"] \arrow[r, phantom, shift right=3, "\gamma\Downarrow"] & b \arrow[r, bend left=50, "k"] \arrow[r, "l" description] \arrow[r, bend right=50, "m"'] \arrow[r, phantom, shift left=3, "\sigma\Downarrow"] \arrow[r, phantom, shift right=3, "\tau\Downarrow"] & c \end{tikzcd}$$

これは

$$
(\tau * \sigma) \bullet (\gamma * \beta) \quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma*\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] \arrow[d, "m"'] \arrow[dr, phantom, "\Downarrow\tau*\sigma"] & b \arrow[d, "k"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dash, dashed] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dash, dashed] & b \arrow[d, equal] \\
b \arrow[rr, dash, dashed] \arrow[d, "m"'] \arrow[drr, phantom, "\Downarrow\tau*\sigma"] & & b \arrow[d, "k"] \\
c \arrow[rr, dash, dashed] & & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dash, dashed] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] \arrow[d, "m"'] \arrow[drr, phantom, "\Downarrow\tau*\sigma"] & b \arrow[r, dash, dashed] & b \arrow[d, "k"] \\
c \arrow[rr, dash, dashed] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dash, dashed] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] \arrow[d, "m"'] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[r, dash, dashed] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "k"] \\
c \arrow[r, dash, dashed] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & c \arrow[r, dash, dashed] & c \arrow[d, equal] \\
c \arrow[rr, dash, dashed] & & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "h"'] & a \arrow[r, dash, dashed] \arrow[d, "g"] & a \arrow[d, "f"] \\
b \arrow[r, phantom, "\Downarrow\tau\bullet\gamma"] \arrow[d, "m"'] & b \arrow[r, phantom, "\Downarrow\sigma\bullet\beta"] \arrow[d, "l"] & b \arrow[d, "k"] \\
c \arrow[r, dash, dashed] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & c \arrow[r, dash, dashed] & c \arrow[d, equal] \\
c \arrow[rr, dash, dashed] & & c
\end{tikzcd}
\quad = \quad (\tau \bullet \gamma) * (\sigma \bullet \beta)
$$

となるから成り立つ．id についても明らかだから M^{abc} は関手である．

- この M は明らかに次の可換図式を満たす．

$$\begin{tikzcd}[column sep=small]
 & \mathcal{V}(\mathbb{D})(c,d)\times\mathcal{V}(\mathbb{D})(b,c)\times\mathcal{V}(\mathbb{D})(a,b) \arrow[dl, "M^{bcd}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times M^{abc}"] & \\
\mathcal{V}(\mathbb{D})(b,d)\times\mathcal{V}(\mathbb{D})(a,b) \arrow[dr, "M^{abd}"'] \arrow[rr, phantom, "="] & & \mathcal{V}(\mathbb{D})(c,d)\times\mathcal{V}(\mathbb{D})(a,c) \arrow[dl, "M^{acd}"] \\
 & \mathcal{V}(\mathbb{D})(a,d) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}\times\mathcal{V}(\mathbb{D})(a,b) \arrow[rr, "\cong"] \arrow[dr, "I^b\times\mathrm{id}"'] & \arrow[d, phantom, "\parallel"] & \mathcal{V}(\mathbb{D})(a,b) \\
 & \mathcal{V}(\mathbb{D})(b,b)\times\mathcal{V}(\mathbb{D})(a,b) \arrow[ur, "M^{abb}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}(\mathbb{D})\times\mathbb{1} \arrow[rr, "\cong"] \arrow[dr, "\mathrm{id}\times I^a"'] & \arrow[d, phantom, "\parallel"] & \mathcal{V}(\mathbb{D})(a,b) \\
 & \mathcal{V}(\mathbb{D})(a,b)\times\mathcal{V}(\mathbb{D})(a,a) \arrow[ur, "M^{aab}"'] &
\end{tikzcd}$$

- 以上により $\mathcal{V}(\mathbb{D})$ は strict 2-category である．

horizontal bicategory については定義することができないが，$a,b\in\mathbb{D}$ に対する Hom 圏 $\mathcal{H}(\mathbb{D})(a,b)$ については次のように定義することができる．

- $\mathcal{H}(\mathbb{D})(a,b)$ の対象は水平射 $J\colon a\nrightarrow b$ とする．
- $\mathcal{H}(\mathbb{D})(a,b)$ の射は次のような cell とする．

$$\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] & b
\end{tikzcd}$$

  但し $\mathrm{dom}(\beta)=J$，$\mathrm{cod}(\beta)=K$ とする．
- 射の合成は $\mathbb{D}$ の合成で定める．
- 恒等射は id_J とする．
- 以上により $\mathcal{H}(\mathbb{D})(a,b)$ は圏である．

augmented virtual double category $\mathbb{D}$ の定義は左右対称になっているので，左右を入れ替えてもやはり augmented virtual double category になる．これを $\mathbb{D}$ の **horizontal dual** という．

2　cartesian cell

以上のように一般化された augmented virtual double category に対しても，pseudo double category と同様の定理が成り立つということをここから第 4 節までで述べる．

まず companion と conjoint を pseudo double category の場合と同様に定義する．

定義. 垂直射 $f\colon a \to b$ の **companion** とは次のような 3 つ組 $\langle f_\star, {}_f\eta, {}_f\varepsilon \rangle$ であって

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"', "\shortmid" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

次の 2 等式を満たすものをいう．

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "f_\star", "\shortmid" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star"', "\shortmid" marking] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star"', "\shortmid" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"', "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

また f の **conjoint** とは次のような 3 つ組 $\langle f^\star, \eta_f, \varepsilon_f \rangle$ であって

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
$$

次の 2 等式を満たすものをいう．

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

次に augumented virtual double category における cartesian cell を以下のように定義する．

定義. $\mathbb{D}$ の cell

$$\begin{tikzcd} a \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, "\vec{K}"', "\shortmid\shortmid"{marking}] & v \end{tikzcd}$$

が **cartesian** $\Longleftrightarrow$ $|\vec{J}| \leq 1$ であり，更に次の右辺のような cell θ に対して，次の等式を満たす cell θ' が一意に存在する．

$$\begin{tikzcd} c \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & x \arrow[d, "k"] \\ a \arrow[r, "\vec{J}"', "\shortmid\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, "\vec{K}"', "\shortmid\shortmid"{marking}] & v \end{tikzcd} \quad = \quad \begin{tikzcd} c \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[d, "h"'] & x \arrow[d, "k"] \\ a \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow\theta"] & u \arrow[d, "g"] \\ b \arrow[r, "\vec{K}"', "\shortmid\shortmid"{marking}] & v \end{tikzcd}$$

$|\vec{J}| = 1$ となるような cartesian cell

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, "\vec{K}"', "\shortmid\shortmid"{marking}] & v \end{tikzcd} \qquad \text{（但し } \vec{J} = \langle a, J, u\rangle \text{ と書いた）}$$

が存在するとき，J を $\vec{K}$ の f と g に沿った**制限**という．普遍性から，このような J は（$\mathcal{H}(\mathbb{D})(a,u)$ において）同型を除いて一意である．そこでこの J を記号では $\vec{K}(f,g)$ と書く．$\vec{K} = \langle b, K, v\rangle$ の場合は $K(f,g)$ のように書く．$\vec{K} = \langle b\rangle$ の場合は $b(f,g)$ のように書く．また cell β も明示するときは制限 $\langle \vec{K}(f,g), \beta\rangle$ という書き方をする．

pseudo double category のときと同様, companion や conjoint は cartesian cell を使って特徴付けることができる．

命題 3. $f\colon a \to b$ の conjoint が存在するとき，ε_f は cartesian である．（horizontal dual を考えれば companion の ${}_f\varepsilon$ も cartesian である．）

証明. 右辺の cell θ を任意に取る．θ' を

$$
\theta' \quad := \quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[dd, "h"'] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uh"] & t \arrow[d, "k"] \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b \arrow[r, "f^\star"', "\shortmid"{marking}] & a
\end{tikzcd}
$$

と定義すると

$$
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & t \arrow[d, "k"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\overset{(\text{定義})}{=}
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[dd, "h"'] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[d, equal] \\
b \arrow[rr, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\varepsilon_f"] & b \arrow[d, "f"] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
\overset{(8),\,(11)}{=}
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[dd, "h"'] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[rr, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
$$

$$
\overset{(12)}{=}
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[dd, "h"'] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] & b \arrow[r, dashed, no head] \arrow[d, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
\overset{(\text{conjoint の定義})}{=}
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortmid\shortmid"{marking}] \arrow[dd, "h"'] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] & b \arrow[r, dashed, no head] \arrow[d, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
$$

$$
\overset{(8),\,(13)}{=}\quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[dd, "h"'] & t \arrow[r, dashed, no head] \arrow[d, "k"'] & t \arrow[d, "k"] \\
\arrow[r, phantom, "\Downarrow\theta"] & a \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow U(f\circ k)"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow U\mathrm{id}"] & b \arrow[r, dashed, no head] & b \arrow[d, equal] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
\quad\overset{(8),\,(13)}{=}\quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[dd, "h"'] & t \arrow[d, "k"] \\
\arrow[r, phantom, "\Downarrow\theta"] & a \arrow[d, "f"'] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[r, phantom, "\Downarrow U\mathrm{id}", shift right=4ex] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad\overset{(8)}{=}\quad \theta
$$

である．一意性を示すため，ψ が

$$
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\psi"] & t \arrow[d, "k"] \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "h"'] & t \arrow[d, "k"] \\
b \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, equal] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

を満たしたとすると

$$
\theta' \quad = \quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[dd, "h"'] & t \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[r, phantom, "\Downarrow Uk", shift right=4ex] & t \arrow[d, "k"] \\
\arrow[r, phantom, "\Downarrow\theta"] & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\psi"] & t \arrow[r, dashed, no head] \arrow[d, "k"] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
s \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\psi"] & t \arrow[r, dashed, no head] \arrow[d, "k"] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] & a \arrow[r, dashed, no head] \arrow[d, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[d, equal] \\
b \arrow[rr, "f^\star"', "\shortmid" marking] & & a
\end{tikzcd}
\quad = \quad \psi
$$

である． □

命題 4. $\mathbb{D}$ の cell

$$
\begin{tikzcd}
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

が cartesian ならば，ある cell

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
b \arrow[r, "J"', "\shortmid"{marking}] & a
\end{tikzcd}
$$

が存在して $\langle J, \gamma, \beta \rangle$ が f の conjoint となる．

証明. β が cartesian だから，ある γ が存在して

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
b \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

とできる．このとき

$$
\begin{tikzcd}
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\beta"] & b \arrow[r, "J"', "\shortmid"{marking}] & a \arrow[d, "f"] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, phantom, "\Downarrow Uf"] \arrow[d, equal] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

$$
= \quad \beta \quad = \quad
\begin{tikzcd}
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & a \arrow[d, equal] \\
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

となるから，cartesian cell β の普遍性により

$$
\begin{tikzcd}
b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b \arrow[r, "J"', "\shortmid"{marking}] & a
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mathrm{id}"] & & a \arrow[d, equal] \\
b \arrow[rr, "J"', "\shortmid"{marking}] & & a
\end{tikzcd}
$$

となり，$\langle J, \gamma, \beta \rangle$ は f の companion である． □

但し pseudo double category とは違って，companion と conjoint が存在するからといって任意の制限が存在するとは限らない．何故かというと pseudo double category のときは合成

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & v \arrow[r, "g^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & u \arrow[d, "g"] \\
b \arrow[r, equal] & b \arrow[r, "J"', "\shortmid"{marking}] & v \arrow[r, equal] & v
\end{tikzcd}
$$

により J の制限を作ることができたが，augmented virtual double category ではこれは cartesian cell にならない．上側の有向パスの長さが 1 以下にならないといけないからである．（逆に言えば上側の水平射 3 つを「合成」することができればこの構成は有効である．補題 12, 14 を参照.）これを踏まえて次の定義をする．

定義. $\mathbb{D}$ を augmented double category とする．

(1) $\mathbb{D}$ が**左制限を持つ**
$\iff$ 任意の水平射 $K\colon b \nrightarrow v$ と垂直射 $f\colon a \to b$ に対して，制限 $K(f, \mathrm{id})$ が存在する．

(2) $\mathbb{D}$ が**右制限を持つ**
$\iff$ 任意の水平射 $K\colon b \nrightarrow v$ と垂直射 $g\colon u \to v$ に対して，制限 $K(\mathrm{id}, g)$ が存在する．

(3) $\mathbb{D}$ が **augmented virtual equipment**
$\iff$ 任意の水平射 $K\colon b \nrightarrow v$ と垂直射 $f\colon a \to b$, $g\colon u \to v$ に対して, 制限 $K(f, g)$ が存在する．

(4) $a \in \mathbb{D}$ が **unital**

$\Longleftrightarrow$ 制限 $a(\mathrm{id},\mathrm{id})$ が存在する．（$I_a \coloneqq a(\mathrm{id},\mathrm{id})$ を a の **horizontal unit** という．）

$$\begin{tikzcd}
a \arrow[r, "I_a", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}$$

(5) $\mathbb{D}$ が **unital** $\Longleftrightarrow$ 任意の対象 $a \in \mathbb{D}$ が unital である．

(6) $\mathbb{D}$ が **virtual equipment** $\Longleftrightarrow$ $\mathbb{D}$ が unital な augmented virtual equipment である．

左制限を持つと言ったときには水平射に対する制限しか考えないため，左制限を持つからといって conjoint が存在するとは限らないが，$\mathbb{D}$ が uinital であれば左制限から conjoint を作ることができる（系 7)．

補題 5. β と γ が次の図式のような cell で，β が cartesian のとき

$$\gamma \text{ が cartesian である} \Longleftrightarrow \beta \circ \gamma \text{ が cartesian である．}$$

$$\begin{tikzcd}
a \arrow[r, "\vec{J}", "||" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & u \arrow[d, "g"] \\
b \arrow[r, "\vec{K}"', "||" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & v \arrow[d, "k"] \\
c \arrow[r, "\vec{H}"', "||" marking] & x
\end{tikzcd}$$

証明. P-cartesian の場合と同様（「fibration」の PDF を参照)． □

命題 6. $f\colon a \to c$ と $g\colon b \to c$ を垂直射として c の horizontal unit $I_c = c(\mathrm{id},\mathrm{id})$ が存在するとする．このとき

$$\text{制限 } I_c(f,g) \text{ が存在する} \Longleftrightarrow \text{制限 } c(f,g) \text{ が存在する．}$$

証明. horizontal unit I_c を与える cartesian cell を β とすると，次の等式により θ と θ'

が 1 対 1 に対応する．

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta'"] & b \arrow[d, "g"] & & a \arrow[r, "J", "\shortmid" marking] \arrow[dd, "f"'] \arrow[ddr, phantom, "\Downarrow\theta"] & b \arrow[dd, "g"] \\
c \arrow[r, "I_c"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[d, equal] & = & & \\
c \arrow[r, dashed, no head] & c & & c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

このとき補題 5 により「制限 $I_c(f,g)$ が存在する（$=\theta'$ が cartesian である）$\Longleftrightarrow$ 制限 $c(f,g)$ が存在する（$=\theta$ が cartesian である）」となる． $\square$

系 7. $\mathbb{D}$ が unital で左制限を持つとき，任意の垂直射の conjoint が存在する． $\square$

命題 8. 次の図式のような cell β を考えたとき

$$\beta \text{ が cartesian である} \Longleftrightarrow \beta \text{ が } \mathcal{H}(\mathbb{D})(a,b) \text{ の同型射である．}$$

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid" marking] & b
\end{tikzcd}
$$

証明. ($\Longrightarrow$) β が cartesian だとして

$$
\begin{tikzcd}
a \arrow[r, "K", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & b \arrow[d, equal] & & a \arrow[r, "K", "\shortmid" marking] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\mathrm{id}_K"] & b \arrow[dd, equal] \\
a \arrow[r, "J"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] & = & & \\
a \arrow[r, "K"', "\shortmid" marking] & b & & a \arrow[r, "K"', "\shortmid" marking] & b
\end{tikzcd}
$$

により γ を取れば，$\mathcal{H}(\mathbb{D})(a,b)$ において $\beta\circ\gamma=\mathrm{id}$ である．よって $\gamma\circ\beta=\mathrm{id}$ を示せば

よい．それは

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & b \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[dd, equal] \\
& \\
a \arrow[r, "K"', "\shortmid"{marking}] & b
\end{tikzcd}
= \beta =
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[dd, equal] \\
& \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

となるから，β の普遍性により

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & b \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\mathrm{id}_J"] & b \arrow[dd, equal] \\
& \\
a \arrow[r, "J"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

である．

($\Longleftarrow$) 明らか． □

3 cocartesian パス

opcartesian cell に相当するものとして，次の弱 cocartesian パスを導入する．

定義． 次のような cell を考える．

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortmid\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[d, "f_1"] \\
b_0 \arrow[r, "\vec{K}_1"', "\shortmid\shortmid"{marking}] & b_1
\end{tikzcd}
, \quad \cdots \quad ,
\begin{tikzcd}
a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid\shortmid"{marking}] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_n \arrow[d, "f_n"] \\
b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid\shortmid"{marking}] & b_n
\end{tikzcd}
$$

(1) $\langle\beta_1, \cdots, \beta_n\rangle$ が **nullary-弱 cocartesian パス**

$\Longleftrightarrow$ 右辺のような cell θ に対して，次の等式を満たす cell θ' が一意に存在する．

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{K}_1"'] \arrow[d, "g"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\theta'"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "h"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[rr, "\vec{J}_1{}^\frown\cdots{}^\frown\vec{J}_n"] \arrow[d, "f_0"'] & & a_n \arrow[d, "f_n"] \\
b_0 \arrow[d, "g"'] & \Downarrow\theta & b_n \arrow[d, "h"] \\
c \arrow[rr, dashed, no head] & & c
\end{tikzcd}
$$

(2) $\langle\beta_1,\cdots,\beta_n\rangle$ が **unary-弱 cocartesian パス**

$\Longleftrightarrow$ 右辺のような cell θ に対して，次の等式を満たす cell θ' が一意に存在する．

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{K}_1"'] \arrow[d, "g"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\theta'"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "h"] \\
c \arrow[rrrr, "H"'] & & & & d
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[rr, "\vec{J}_1{}^\frown\cdots{}^\frown\vec{J}_n"] \arrow[d, "f_0"'] & & a_n \arrow[d, "f_n"] \\
b_0 \arrow[d, "g"'] & \Downarrow\theta & b_n \arrow[d, "h"] \\
c \arrow[rr, "H"'] & & d
\end{tikzcd}
$$

(3) $\langle\beta_1,\cdots,\beta_n\rangle$ が**弱 cocartesian パス**

$\Longleftrightarrow$ $\langle\beta_1,\cdots,\beta_n\rangle$ が nullary-弱 cocartesian パスかつ unary-弱 cocartesian パスである．

また $\langle\beta\rangle$ が弱 cocartesian パスになるような cell β を**弱 cocartesian cell** という．**nullary-弱 cocartesian cell** や **unary-弱 cocartesian cell** も同様である．

命題 9. $\langle\beta_1,\cdots,\beta_n\rangle$ を弱 cocartesian パスとする．このときこれに Uf_i をいくつか入れた $\langle Uf_0,\beta_1,\cdots,\beta_i,Uf_i,\beta_{i+1},\cdots\beta_n,Uf_n\rangle$ も弱 cocartesian パスである．

証明. augmented virtual double category の条件 (12) から明らか． □

つまり弱 cocartesian パスを考えるときには Uf の有無については気にしなくてよい（これは以下の左 cocartesian パス等でも同様である）．

pseudo double category における opcartesian cell では，β が cartesian で γ が opcartesian ならば

$$
\begin{tikzcd}
a \arrow[r, "J"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "M"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[d, "f"] \\
b \arrow[r, "K"'] & v \arrow[r, "N"'] & x
\end{tikzcd}
$$

も opcartesian であった．ところが弱 cocartesian パスではこれは成り立たない．そこで次のように cocartesian パスを定義する．

定義. nullary-弱 cocartesian パス $\langle \beta_1, \cdots, \beta_n \rangle$ が **nullary-左 cocartesian パス**
$\Longleftrightarrow m \geq 1$，$\vec{H} = \langle c_0, H_1, c_1, \cdots, c_{m-1}, H_m, b_0 \rangle$ を有向パスとして制限 $\langle H_m(\mathrm{id}, f_0), \gamma \rangle$ が存在するとする．このとき $\langle \mathrm{id}_{H_1}, \cdots, \mathrm{id}_{H_{m-1}}, \gamma, \beta_1, \cdots, \beta_n \rangle$ が nullary-弱 cocartesian パスである．（つまり次のような状況である．）

$$\begin{tikzcd}
c_0 \arrow[r, "H_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H_1}"] & c_1 \arrow[r, "H_2"] \arrow[d, equal] & \cdots \arrow[r, "H_{m-2}"] \arrow[d, phantom, "\cdots"] & c_{m-2} \arrow[r, "H_{m-1}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H_{m-1}}"] & c_{m-1} \arrow[r, "H_m(\mathrm{id}, f_0)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \gamma"] & a_0 \arrow[r, "\vec{J}_1"] \arrow[d, "f_0"] \arrow[dr, phantom, "\Downarrow \beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"] \arrow[dr, phantom, "\Downarrow \beta_n"] & a_n \arrow[d, "f_n"] \\
c_0 \arrow[r, "H_1"'] & c_1 \arrow[r, "H_2"'] & \cdots \arrow[r, "H_{m-2}"'] & c_{m-2} \arrow[r, "H_{m-1}"'] & c_{m-1} \arrow[r, "H_m"'] & b_0 \arrow[r, "\vec{K}_1"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n
\end{tikzcd}$$

同様にして **unary-左 cocartesian パス**も定義する．nullary-左 cocartesian かつ unary-左 cocartesian のとき**左 cocartesian パス**という．

また horizontal dual を考えることで**右 cocartesian パス**などを定義する．

定義. nullary-左 cocartesian パス $\langle \beta_1, \cdots, \beta_n \rangle$ が **nullary-cocartesian パス**
$\Longleftrightarrow \vec{H} = \langle c_0, H_1, c_1, \cdots, c_{m-1}, H_m, b_0 \rangle$ をパスとして制限 $\langle H_m(\mathrm{id}, f_0), \gamma \rangle$ が存在するとする．このとき $\langle \mathrm{id}_{H_1}, \cdots, \mathrm{id}_{H_{m-1}}, \gamma, \beta_1, \cdots, \beta_n \rangle$ が nullary-右 cocartesian パスである．

同様にして **unary-cocartesian パス**も定義する．nullary-cocartesian かつ unary-cocartesian のとき **cocartesian パス**という．

つまり $\langle \beta_1, \cdots, \beta_n \rangle$ が cocartesian パスとは，$\vec{H} = \langle c_0, H_1, c_1, \cdots, c_{m-1}, H_m, b_0 \rangle$，$\vec{H}' = \langle b_n, H'_1, d_1, \cdots, d_{l-1}, H'_l, d_l \rangle$ として制限 $\langle H_m(\mathrm{id}, f_0), \gamma \rangle$ と $\langle H'_1(f_n, \mathrm{id}), \sigma \rangle$ が存在するならば

$$\begin{tikzcd}
c_0 \arrow[r, "H_1"] \arrow[d, equal] & \cdots \arrow[r, "H_{m-1}"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & c_{m-1} \arrow[r, "H_m(\mathrm{id}, f_0)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \gamma"] & a_0 \arrow[r, "\vec{J}_1"] \arrow[d, "f_0"] \arrow[dr, phantom, "\Downarrow \beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"] \arrow[dr, phantom, "\Downarrow \beta_n"] & a_n \arrow[r, "H'_1(f_n, \mathrm{id})"] \arrow[d, "f_n"] \arrow[dr, phantom, "\Downarrow \sigma"] & d_1 \arrow[r, "H'_2"] \arrow[d, equal] & \cdots \arrow[r, "H'_l"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & d_l \arrow[d, equal] \\
c_0 \arrow[r, "H_1"'] & \cdots \arrow[r, "H_{m-1}"'] & c_{m-1} \arrow[r, "H_m"'] & b_0 \arrow[r, "\vec{K}_1"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[r, "H'_1"'] & d_1 \arrow[r, "H'_2"'] & \cdots \arrow[r, "H'_l"'] & d_l
\end{tikzcd}$$

が弱 cocartesian パスになるということである．（但し

$$
\begin{tikzcd}
c_0 \arrow[r, "H_1", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H_1}"] & c_1 \arrow[r, "H_2", "|" marking] \arrow[d, equal] & \cdots \arrow[r, "H_{m-2}", "|" marking] \arrow[d, phantom, "\cdots"] & c_{m-2} \arrow[r, "H_{m-1}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H_{m-1}}"] & c_{m-1} \arrow[d, equal] \\
c_0 \arrow[r, "H_1"', "|" marking] & c_1 \arrow[r, "H_2"', "|" marking] & \cdots \arrow[r, "H_{m-2}"', "|" marking] & c_{m-2} \arrow[r, "H_{m-1}"', "|" marking] & c_{m-1}
\end{tikzcd}
$$

を単に

$$
\begin{tikzcd}
c_0 \arrow[r, "H_1", "|" marking] \arrow[d, equal] & \cdots \arrow[r, "H_{m-1}", "|" marking] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & c_{m-1} \arrow[d, equal] \\
c_0 \arrow[r, "H_1"', "|" marking] & \cdots \arrow[r, "H_{m-1}"', "|" marking] & c_{m-1}
\end{tikzcd}
$$

と略記した．以下同様の略記を行う．）

定義から明らかに次のような関係が成り立つ．（但し，例えば左 cocartesian かつ右 cocartesian だからといって cocartesian とは限らないことに注意する．）

$$
\begin{tikzcd}[column sep=small]
\text{nullary-左 cocartesian} \arrow[rr] & & \text{nullary-弱 cocartesian} & \\
& \text{nullary-cocartesian} \arrow[rr] \arrow[ul] & & \text{nullary-右 cocartesian} \arrow[ul] \\
\text{左 cocartesian} \arrow[rr] \arrow[uu] \arrow[dd] & & \text{弱 cocartesian} \arrow[uu] \arrow[dd] & \\
& \text{cocartesian} \arrow[rr] \arrow[ul] \arrow[uu] \arrow[dd] & & \text{右 cocartesian} \arrow[ul] \arrow[uu] \arrow[dd] \\
\text{unary-左 cocartesian} \arrow[rr] & & \text{unary-弱 cocartesian} & \\
& \text{unary-cocartesian} \arrow[rr] \arrow[ul] & & \text{unary-右 cocartesian} \arrow[ul]
\end{tikzcd}
$$

cocartesian パスの垂直合成については次のようになる．

補題 10. $1 \leq i \leq n$ に対して次のような cell を考える.

$$\varphi^i \quad := \quad \begin{tikzcd}
a^i_0 \arrow[r, "\vec{J}^i_1", "\shortmid"{marking}] \arrow[d, "f^i_0"'] \arrow[dr, phantom, "\Downarrow\beta^i_1"] & a^i_1 \arrow[r, "\vec{J}^i_2", "\shortmid"{marking}] \arrow[d, "f^i_1"] \arrow[dr, phantom, "\cdots"] & \cdots \arrow[r, "\vec{J}^i_{n^i-1}", "\shortmid"{marking}] & a^i_{n^i-1} \arrow[r, "\vec{J}^i_{n^i}", "\shortmid"{marking}] \arrow[d, "f^i_{n^i-1}"'] \arrow[dr, phantom, "\Downarrow\beta^i_{n^i}"] & a^i_{n^i} \arrow[d, "f^i_{n^i}"] \\
b^i_0 \arrow[r, "\vec{K}^i_1"', "\shortmid"{marking}] \arrow[d, "g^i"'] & b^i_1 \arrow[r, "\vec{K}^i_2"', "\shortmid"{marking}] & \cdots \arrow[r, "\vec{K}^i_{n^i-1}"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma^i"] & b^i_{n^i-1} \arrow[r, "\vec{K}^i_{n^i}"', "\shortmid"{marking}] & b^i_{n^i} \arrow[d, "h^i"] \\
c^{i-1} \arrow[rrrr, "\vec{L}^i"', "\shortmid"{marking}] & & & & c^i
\end{tikzcd}$$

$f^i_{n^i} = f^{i+1}_0$, $h^i = g^{i+1}$ であるとする（即ちパス $\langle \varphi^1, \cdots, \varphi^n \rangle$ が考えられるということ）. 更に $\langle \beta^1_1, \cdots, \beta^n_{n^n} \rangle$ は弱 cocartesian であるとする．このとき

$$\langle \gamma^1, \cdots, \gamma^n \rangle \text{ が弱 cocartesian パス} \iff \langle \varphi^1, \cdots, \varphi^n \rangle \text{ が弱 cocartesian パス.}$$

また同様の同値が nullary-弱 cocartesian パスと unary-弱 cocartesian パスについても成り立つ.

証明. 補題 5 と同様に分かる. □

補題 11. 補題 10 において $\langle \beta^1_1, \cdots, \beta^n_{n^n} \rangle$ が左 cocartesian パスであるとする.

(1) 任意の $K\colon d \nrightarrow c^0$ に対して「制限 $K(\mathrm{id}_d, g^1 \circ f^1_0)$ が存在するならば $K(\mathrm{id}_d, g^1)$ も存在する」が成り立つとする．このとき $\langle \gamma^1, \cdots, \gamma^n \rangle$ が左 cocartesian パスならば，$\langle \varphi^1, \cdots, \varphi^n \rangle$ も左 cocartesian パスである.

(2) 任意の $K\colon d \nrightarrow c^0$ に対して「制限 $K(\mathrm{id}_d, g^1)$ が存在するならば $K(\mathrm{id}_d, g^1 \circ f^1_0)$ も存在する」が成り立つとする．このとき $\langle \varphi^1, \cdots, \varphi^n \rangle$ が左 cocartesian パスならば，$\langle \gamma^1, \cdots, \gamma^n \rangle$ も左 cocartesian パスである.

また同様のことが nullary-左 cocartesian パスと unary-左 cocartesian パスについても成り立つ．horizontal dual を考えれば右についても成り立つ．それらを組み合わせることで同様のことが nullary-cocartesian パス，unary-cocartesian パス，cocartesian パスに対しても言える.

証明. 補題 5 と同様に分かる. □

$f^1_0 = \mathrm{id}$ の場合，補題 11 の制限に関する条件は自明になるから，この場合は常に

$$\langle \gamma^1, \cdots, \gamma^n \rangle \text{ が左 cocartesian} \iff \langle \varphi^1, \cdots, \varphi^n \rangle \text{ が左 cocartesian}$$

となる．また $\mathbb{D}$ が右制限を持つ場合も同様のことが言える．

右 cocartesian や cocartesian についても同様である．

定義. virtual double category $\mathbb{D}$ の有向パス $\vec{J}$ が**合成可能**
$\Longleftrightarrow$ 次の図式のような水平射 K と cocartesian cell β が存在する．

$$\begin{tikzcd} a \arrow[r, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\ a \arrow[r, "K"'] & b \end{tikzcd}$$

またこのとき K を $\vec{J}$ の**合成**という．$\vec{J} = \langle a, J_1, \cdots, J_n, b\rangle$ のとき，その合成を記号で $J_1 \odot \cdots \odot J_n$ と書く．

ここで次の図式を考える

$$\begin{tikzcd} a \arrow[d, "f"'] & c \arrow[d, "g"] \\ b \arrow[r, "\vec{J}"'] & d \end{tikzcd}$$

$|\vec{J}| \leq 1$ として，更に companion $f_\star$ と conjoint $g^\star$ が存在するとする．

補題 12. 制限 $\langle \vec{J}(f,g), \beta\rangle$ が存在するならば $f_\star \frown \vec{J} \frown g^\star$ は合成可能[*7]である．

証明. β の普遍性により次の γ を得る．

$$\begin{tikzcd} a \arrow[r, "f_\star"] \arrow[d, equal] & b \arrow[r, "\vec{J}"] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"] & c \arrow[d, equal] \\ a \arrow[rrr, "\vec{J}(f,g)"'] \arrow[d, "f"'] & & & c \arrow[d, "g"] \\ b \arrow[rrr, "\vec{J}"'] & \arrow[u, phantom, "\Downarrow\beta"] & & d \end{tikzcd} = \begin{tikzcd} a \arrow[r, "f_\star"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow f^\varepsilon"] & b \arrow[r, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_g"] & c \arrow[d, "g"] \\ b \arrow[r, dashed, no head] & b \arrow[r, "\vec{J}"'] & d \arrow[r, dashed, no head] & d \end{tikzcd}$$

[*7] $\langle a, f_\star, b\rangle \frown \vec{J} \frown \langle v, g^\star, u\rangle$ を単に $f_\star \frown \vec{J} \frown g^\star$ と書いている．

(但し $|\vec{J}|=0$ の場合はこれは

$$
\begin{tikzcd}
a \arrow[r, "f_\star"] \arrow[d, equal] & b \arrow[r, "g^\star"] \arrow[d, phantom, "\Downarrow\gamma"] & c \arrow[d, equal] \\
a \arrow[rr, "\vec{J}(f,g)"'] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "f_\star"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[r, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

という意味である．以下同様とする．）この γ が cocartesian であることを示せばよい．まず準備として

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "\vec{J}(f,g)"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & d \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] & b \arrow[r, "\vec{J}"'] & d \arrow[r, "g^\star"'] & d \arrow[d, equal] \\
a \arrow[rrr, "\vec{J}(f,g)"'] & \arrow[u, phantom, shift left=8, "\Downarrow\gamma"] & & d
\end{tikzcd}
\quad = \quad \mathrm{id} \tag{13}
$$

を示す．

∵) 定義より

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "\vec{J}(f,g)"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] & b \arrow[r, "\vec{J}"'] & d \arrow[r, "g^\star"'] & c \arrow[d, equal] \\
a \arrow[rrr, "\vec{J}(f,g)"'] \arrow[d, "f"'] & \arrow[u, phantom, shift left=8, "\Downarrow\gamma"] & & c \arrow[d, "g"] \\
b \arrow[rrr, "\vec{J}"'] & \arrow[u, phantom, shift left=8, "\Downarrow\beta"] & & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "\vec{J}(f,g)"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[r, "\vec{J}"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, "\vec{J}"'] & d \arrow[r, dashed, no head] & d
\end{tikzcd}
\quad = \quad \beta
$$

となる．よって β が cartesian だから (13) が分かる．

これを使って γ が cocartesian であることを示すが，どの場合も同様であるから，ここでは $\langle \mathrm{id}_H, \gamma, \mathrm{id}_{H'} \rangle$ が弱 cocartesian パスであることのみを示す．そのために次の図式の

cell θ を任意に取る．

$$
\theta' \quad := \quad
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & a \arrow[r, "\vec{J}(f,g)"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[r, "f_\star"'] & b \arrow[r, "\vec{J}"'] \arrow[d, phantom, "\Downarrow\theta"] & d \arrow[r, "g^\star"'] & c \arrow[r, "H'"'] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{K}"'] & & & & & z'
\end{tikzcd}
$$

上記のように θ' を定義すると

$$
\begin{tikzcd}
c \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, "f_\star{}^\frown\vec{J}{}^\frown g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & u \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & d \arrow[d, equal] \\
c \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[r, "\vec{J}(f,g)"'] \arrow[d, phantom, "\Downarrow\theta'"] & u \arrow[r, "H'"'] & d \arrow[d, "h"] \\
c' \arrow[rrr, "\vec{K}"'] & & & d'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, "f_\star"] \arrow[d, equal] & b \arrow[r, "\vec{J}"] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & a \arrow[r, "\vec{J}(f,g)"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[r, "f_\star"'] & b \arrow[r, "\vec{J}"'] \arrow[d, phantom, "\Downarrow\theta"] & d \arrow[r, "g^\star"'] & c \arrow[r, "H'"'] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{K}"'] & & & & & z'
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, "f_\star"] \arrow[d, equal] & b \arrow[r, "\vec{J}"] & d \arrow[r, "g^\star"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[d, equal] & & \Downarrow\mathrm{id} & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[r, "f_\star"'] & b \arrow[r, "\vec{J}"'] \arrow[d, phantom, "\Downarrow\theta"] & d \arrow[r, "g^\star"'] & c \arrow[r, "H'"'] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{K}"'] & & & & & z'
\end{tikzcd}
\quad = \quad \theta
$$

である．このような θ' が一意であることを示すため θ'' が

$$
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, "f_\star{}^\frown\vec{J}{}^\frown g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[r, "\vec{J}(f,g)"'] \arrow[dr, phantom, "\Downarrow\theta''"] & c \arrow[r, "H'"'] & z \arrow[d, "h"] \\
x' \arrow[rrr, "\vec{K}"'] & & & z'
\end{tikzcd}
\quad = \quad \theta
$$

を満たしたとすると

$$
\theta' \quad = \quad
\begin{tikzcd}
x \arrow[r, "H", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_H"] & a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & a \arrow[r, "\vec{J}(f,g)", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow \beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow \varepsilon_g"] & c \arrow[r, "H'", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"', "|" marking] \arrow[d, "k"'] & a \arrow[r, "f_\star"', "|" marking] & b \arrow[r, "\vec{J}"', "||" marking] \arrow[d, phantom, "\Downarrow \theta"] & d \arrow[r, "g^\star"', "|" marking] & c \arrow[r, "H'"', "|" marking] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{K}"', "||" marking] & & & & & z'
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
x \arrow[r, "H", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_H"] & a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & a \arrow[r, "\vec{J}(f,g)", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow \beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow \varepsilon_g"] & c \arrow[r, "H'", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_H"] & a \arrow[r, "f_\star"', "|" marking] \arrow[d, equal] & b \arrow[r, "\vec{J}"', "||" marking] \arrow[d, phantom, "\Downarrow \gamma"] & d \arrow[r, "g^\star"', "|" marking] & c \arrow[r, "H'"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"', "|" marking] \arrow[d, "k"'] & a \arrow[rrr, "\vec{J}(f,g)"', "|" marking] & \arrow[d, phantom, "\Downarrow \theta''"] & & c \arrow[r, "H'"', "|" marking] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{K}"', "||" marking] & & & & & z'
\end{tikzcd}
\quad \overset{(13)}{=} \quad \theta''
$$

である. $\square$

補題 14. $f_\star \frown \vec{J} \frown g^\star$ が合成可能ならば制限 $\vec{J}(f,g)$ が存在する.

証明. 次の左辺のような cocartesian cell γ が存在するとして

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "|" marking] \arrow[d, equal] & b \arrow[r, "\vec{J}", "||" marking] \arrow[d, phantom, "\Downarrow \gamma"] & d \arrow[r, "g^\star", "|" marking] & c \arrow[d, equal] \\
a \arrow[rrr, "K"', "|" marking] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow \beta"] & & c \arrow[d, "g"] \\
b \arrow[rrr, "\vec{J}"', "||" marking] & & & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "f_\star", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[r, "\vec{J}", "||" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, "\vec{J}"', "||" marking] & d \arrow[r, dashed, no head] & d
\end{tikzcd}
$$

により β を得る．この β が cartesian であることを示せばよい．まず

$$
\begin{tikzcd}
a \arrow[r, dashed, -] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[r, "K", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, -] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"', "|" marking] \arrow[d, equal] & b \arrow[r, "\vec{J}"', "||" marking] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"', "|" marking] & c \arrow[d, equal] \\
a \arrow[rrr, "K"', "|" marking] & & & c
\end{tikzcd}
\quad = \quad \mathrm{id} \tag{15}
$$

である．

∵) 定義より

$$
\begin{tikzcd}
a \arrow[r, dashed, -] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[r, "f_\star{}^\frown\vec{J}{}^\frown g^\star", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & c \arrow[r, dashed, -] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & c \arrow[d, equal] \\
a \arrow[r, dashed, -] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[r, "K", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, -] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"', "|" marking] \arrow[d, equal] & b \arrow[r, "\vec{J}"', "||" marking] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"', "|" marking] & c \arrow[d, equal] \\
a \arrow[rrr, "K"', "|" marking] & & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, -] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[r, "f_\star", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow_f\varepsilon"] & b \arrow[r, "\vec{J}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[r, dashed, -] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"', "|" marking] \arrow[d, equal] & b \arrow[r, dashed, -] & b \arrow[r, "\vec{J}"', "||" marking] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, dashed, -] & d \arrow[r, "g^\star"', "|" marking] & c \arrow[d, equal] \\
a \arrow[rrrrr, "K"', "|" marking] & & & & & c
\end{tikzcd}
$$

$$
= \quad \gamma
$$

となるから γ が cocartesian（と命題 9）より分かる．

β が cartesian であることを示すため，次の右辺の θ を取る．このとき

$$
\theta' \quad := \quad
\begin{tikzcd}
x \arrow[r, dashed, -] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & x \arrow[r, "\vec{H}", "||" marking] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\theta"] & z \arrow[r, dashed, -] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & z \arrow[d, "k"] \\
a \arrow[r, dashed, -] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[d, "f"] & c \arrow[r, dashed, -] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"', "|" marking] \arrow[d, equal] & b \arrow[r, "\vec{J}"', "||" marking] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"', "|" marking] & c \arrow[d, equal] \\
a \arrow[rrr, "K"', "|" marking] & & & c
\end{tikzcd}
$$

と定義すると

$$
\begin{tikzcd}
x \arrow[r, "\vec{H}"] \arrow[dd, "h"'] \arrow[ddr, phantom, "\Downarrow\theta'"] & z \arrow[dd, "k"] \\
& \\
a \arrow[r, "K"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[r, "\vec{J}"'] & d
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & x \arrow[r, "\vec{H}"] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\theta"] & z \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & z \arrow[d, "k"] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] & c \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\gamma"] & b \arrow[r, "\vec{J}"'] & d \arrow[r, "g^\star"'] & c \arrow[d, equal] \\
a \arrow[rrr, "K"'] \arrow[d, "f"'] \arrow[drrr, phantom, "\Downarrow\beta"] & & & c \arrow[d, "g"] \\
b \arrow[rrr, "\vec{J}"'] & & & d
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & x \arrow[r, "\vec{H}"] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\theta"] & z \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & z \arrow[d, "k"] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] & c \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[r, "\vec{J}"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, "\vec{J}"'] & d \arrow[r, dashed, no head] & d
\end{tikzcd}
$$

$$= \quad \theta$$

である．このような θ' が一意であることを示すため θ'' が

$$
\begin{tikzcd}
x \arrow[r, "\vec{H}"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta''"] & z \arrow[d, "k"] \\
a \arrow[r, "K"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[r, "\vec{J}"'] & d
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "\vec{H}"] \arrow[d, "h"'] & z \arrow[d, "k"] \\
a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & c \arrow[d, "g"] \\
b \arrow[r, "\vec{J}"'] & d
\end{tikzcd}
$$

を満たしたとすると

$$
\theta' =
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & x \arrow[r, "\vec{H}"] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\theta"] & z \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & z \arrow[d, "k"] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] & c \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\gamma"] & b \arrow[r, "\vec{J}"'] & d \arrow[r, "g^\star"'] & c \arrow[d, equal] \\
a \arrow[rrr, "K"'] & & & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & x \arrow[r, "\vec{H}"] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\theta''"] & z \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & z \arrow[d, "k"] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "K"'] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\gamma"] & b \arrow[r, "\vec{J}"'] & d \arrow[r, "g^\star"'] & c \arrow[d, equal] \\
a \arrow[rrr, "K"'] & & & c
\end{tikzcd}
\overset{(15)}{=} \theta''
$$

である． □

※ 証明から分かる通り，実は γ が弱 cocartesian であれば制限 $\langle \vec{J}(f,g), \beta\rangle$ が得られる．従って補題 12 により，この場合の γ は弱 cocartesian ならば cocartesian である．

補題 16. β と γ が等式

$$
\begin{tikzcd}
a \arrow[r, "f_\star"] \arrow[d, equal] & b \arrow[r, "\vec{J}"] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"] & c \arrow[d, equal] \\
a \arrow[rrr, "K"'] \arrow[d, "f"'] & & & c \arrow[d, "g"] \\
b \arrow[rrr, "\vec{J}"'] & \arrow[u, phantom, "\Downarrow\beta"] & & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "f_\star"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow_f\varepsilon"] & b \arrow[r, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, "\vec{J}"'] & d \arrow[r, dashed, no head] & d
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[r, "K"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] & b \arrow[r, "\vec{J}"'] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"'] & c \arrow[d, equal] \\
a \arrow[rrr, "K"'] & & & c
\end{tikzcd}
\quad = \quad \mathrm{id}
$$

を満たすならば $\langle {}_f\eta, \beta, \eta_f\rangle$ は cocartesian パスである．（特に補題 12, 14 における $\langle {}_f\eta, \beta, \eta_f\rangle$ は cocartesian パスである．）

証明. どの場合も同様であるから，ここでは $\langle \mathrm{id}_H, {}_f\eta, \beta, \eta_f, \mathrm{id}_{H'}\rangle$ が弱 cocartesian パスであることのみを示す．そのために次の右辺の θ が与えられたときに θ' を

$$
\theta' \quad := \quad
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & a \arrow[r, "f_\star"] \arrow[d, equal] & b \arrow[r, "\vec{J}"] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{H'}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[rrr, "K"'] & & & c \arrow[r, "H'"'] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{H}''"'] & & \arrow[u, phantom, "\Downarrow\theta"] & & & z'
\end{tikzcd}
$$

と定義すると

$$
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & a \arrow[r, "K"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dash, dashed] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[dd, "k"'] & a \arrow[r, "f_\star"'] & b \arrow[r, "\vec{J}"'] & d \arrow[r, "g^\star"'] & c \arrow[r, "H'"'] & z \arrow[dd, "h"] \\
 & & & \Downarrow\theta' & & \\
x' \arrow[rrrrr, "\vec{H}''"'] & & & & & z'
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
x \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & a \arrow[r, "K"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, dash, dashed] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[r, "H'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, "f_\star"'] \arrow[d, equal] & b \arrow[r, "\vec{J}"'] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"'] & c \arrow[r, "H'"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & z \arrow[d, equal] \\
x \arrow[r, "H"'] \arrow[d, "k"'] & a \arrow[rrr, "K"'] & \Downarrow\theta & & c \arrow[r, "H'"'] & z \arrow[d, "h"] \\
x' \arrow[rrrrr, "\vec{H}''"'] & & & & & z'
\end{tikzcd}
\quad = \quad \theta
$$

となる．もう 1 つの等式から θ' の一意性も分かり $\langle \mathrm{id}_H, {}_f\eta, \beta, \eta_f, \mathrm{id}_{H'} \rangle$ は弱 cocartesian パスである． □

$f = \mathrm{id}$ の場合は，上記と同様のことが ${}_f\eta, {}_f\varepsilon$ を除いた形で成り立つ．つまり以下のようになる．

まず制限 $\langle \vec{J}(\mathrm{id}, g), \beta \rangle$ が存在するならば $\vec{J} \frown g^\star$ は合成可能である．実際

$$
\begin{tikzcd}
b \arrow[r, "\vec{J}"] \arrow[d, equal] & d \arrow[r, "g^\star"] \arrow[d, phantom, "\Downarrow\gamma"] & c \arrow[d, equal] \\
b \arrow[rr, "\vec{J}(\mathrm{id},g)"'] \arrow[d, equal] & \Downarrow\beta & c \arrow[d, "g"] \\
b \arrow[rr, "\vec{J}"'] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "\vec{J}"] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[dd, "g"] \\
 & & \\
b \arrow[r, dash, dashed] & d \arrow[r, "\vec{J}"'] & c
\end{tikzcd}
$$

とすればよい．逆に $\vec{J} \frown g^\star$ が合成可能な場合も，この等式により制限 $\langle \vec{J}(\mathrm{id}, g), \beta \rangle$ が得

られる．また β と γ が等式

$$
\begin{tikzcd}
b \arrow[r, "\vec{J}"] \arrow[d, equal] & d \arrow[r, "g^\star"] \arrow[d, phantom, "\Downarrow\gamma"] & c \arrow[d, equal] \\
b \arrow[rr, "K"'] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[rr, "\vec{J}"'] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dash, dashed] & d \arrow[r, "\vec{J}"'] & c
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, "K"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dash, dashed] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\eta_g"] & c \arrow[d, equal] \\
a \arrow[r, "\vec{J}"'] \arrow[d, equal] & b \arrow[r, "g^\star"'] \arrow[d, phantom, "\Downarrow\gamma"] & c \arrow[d, equal] \\
a \arrow[rr, "K"'] & & c
\end{tikzcd}
\quad = \quad \mathrm{id}
$$

を満たすならば $\langle \eta_f, \beta \rangle$ は cocartesian パスである．

$g = \mathrm{id}$ の場合も同様である．

命題 17. $f\colon a \to b$ の conjoint が存在するとき，η_f は cocartesian cell である．（horizontal dual を考えれば companion の ${}_f\eta$ も cocartesian cell である．）

証明. まず η_f が右 cocartesian であることを示す．どの場合も同様なので，ここでは β が cartesian のときに

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"'] & a
\end{tikzcd},
\quad
\begin{tikzcd}
a \arrow[r, "J(\mathrm{id},\mathrm{id})"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[d, equal] \\
a \arrow[r, "J"'] & c
\end{tikzcd},
\quad
\begin{tikzcd}
c \arrow[r, "H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & d \arrow[d, equal] \\
c \arrow[r, "H"'] & d
\end{tikzcd}
$$

が弱 cocartesian であることのみを示す．そのために次の図式の cell θ を任意に取る．β

が cartesian だから命題 8 より β は同型である．そこで θ' を

$$
\theta' \quad := \quad
\begin{tikzcd}
b \arrow[r, "f^\star", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[r, "J", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & c \arrow[r, "H", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & d \arrow[d, equal] \\
b \arrow[r, "f^\star"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, "{J(\mathrm{id},\mathrm{id})}"', "|"{marking}] \arrow[d, "f"] & c \arrow[r, "H"', "|"{marking}] \arrow[d, phantom, "\Downarrow\theta"] & d \arrow[dd, "h"] \\
b \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & b \arrow[d, "k"] & & \\
u \arrow[r, dash, dashed] & u \arrow[rr, "\vec{L}"', "|"{marking}] & & v
\end{tikzcd}
$$

により定義すれば

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, "{J(\mathrm{id},\mathrm{id})}", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & c \arrow[r, "H", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & d \arrow[d, equal] \\
b \arrow[r, "f^\star"', "|"{marking}] \arrow[d, "k"'] & a \arrow[r, "J"', "|"{marking}] \arrow[d, phantom, "\Downarrow\theta'"] & c \arrow[r, "H"', "|"{marking}] & d \arrow[d, "h"] \\
u \arrow[rrr, "\vec{L}"', "|"{marking}] & & & v
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, "{J(\mathrm{id},\mathrm{id})}", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[r, "H", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & d \arrow[d, equal] \\
b \arrow[r, "f^\star"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[r, "J"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & c \arrow[r, "H"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_H"] & d \arrow[d, equal] \\
b \arrow[r, "f^\star"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, "{J(\mathrm{id},\mathrm{id})}"', "|"{marking}] \arrow[d, "f"] & c \arrow[r, "H"', "|"{marking}] \arrow[d, phantom, "\Downarrow\theta"] & d \arrow[dd, "h"] \\
b \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & b \arrow[d, "k"] & & \\
u \arrow[r, dash, dashed] & u \arrow[rr, "\vec{L}"', "|"{marking}] & & v
\end{tikzcd}
\quad = \quad \theta
$$

となる．

後は次の β が cartesian のとき $\langle \beta, \eta_f \rangle$ が cocartesian パスであることを示せばよい．

$$
\begin{tikzcd}
c \arrow[r, "{J(\mathrm{id},f)}", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
c \arrow[r, "J"', "|"{marking}] & b \arrow[r, "f^\star"', "|"{marking}] & a
\end{tikzcd}
$$

これは補題 16 の直後に述べたことにより成り立つ． □

4 Kan 拡張

4.1 Kan 拡張

augumented virtual double category における Kan 拡張は次のようになる.

定義. augmented virtual double category $\mathbb{D}$ の cell

$$\begin{tikzcd} a \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \end{tikzcd}$$

が $\vec{J}$ に沿った g の**左 Kan 拡張** $\Longleftrightarrow$ 右辺のような cell θ に対して，左辺のような τ が一意に存在して

$$\begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & a \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c \end{tikzcd} = \begin{tikzcd} a \arrow[rr, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "k"'] \arrow[drr, phantom, "\Downarrow\theta"] & & b \arrow[d, "g"] \\ c \arrow[rr, dashed, no head] & & c \end{tikzcd}$$

となる.

左 Kan 拡張は普遍性で定義しているから，$\vec{J}$ に沿った g の左 Kan 拡張

$$\begin{tikzcd} a \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \end{tikzcd}$$

が存在するならば，この l は（$\mathcal{V}(\mathbb{D})(a,c)$ における）同型を除いて一意である．この l を $\vec{J}^{\dagger}g$ で表す.

pseudo double category のときと同様，左 Kan 拡張は 2-category における左 Kan 拡張の一般化である（定理 19).

補題 18. 次の右辺の図式において $\langle\beta_1,\cdots,\beta_n\rangle$ が nullary-弱 cocartesian パスのとき，κ

を

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, phantom, "\Downarrow\kappa"] \arrow[d, "l"'] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & a \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, "\shortmid" marking, "\vec{J}_1"'] \arrow[d, "l"'] & b_1 \arrow[r, "\shortmid" marking, "\vec{J}_2"'] & \cdots \arrow[r, "\shortmid" marking, "\vec{J}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\shortmid" marking, "\vec{J}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

により定義すると

$$\mathcal{V}(\mathbb{D})\text{ において } f_0^{\dagger}(g\circ f_n)=\langle l,\kappa\rangle\text{ である} \iff \eta\text{ が左 Kan 拡張である.}$$

証明. $\langle\beta_1,\cdots,\beta_n\rangle$ が nullary-弱 cocartesian だから等式

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "k"'] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & a \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, "\shortmid" marking, "\vec{J}_1"'] \arrow[d, "k"'] & b_1 \arrow[r, "\shortmid" marking, "\vec{J}_2"'] & \cdots \arrow[r, "\shortmid" marking, "\vec{J}_{n-1}"'] \arrow[d, phantom, "\Downarrow\theta'"] & b_{n-1} \arrow[r, "\shortmid" marking, "\vec{J}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

により θ と θ' は 1 対 1 に対応する．このとき $\mathcal{V}(\mathbb{D})$ の 2-morphism $\tau\colon l\Rightarrow k$ に対して

$$
\begin{tikzcd}
b_0 \arrow[dr, "l"'] \arrow[dr, bend left=40, "k"] \arrow[dr, phantom, bend left=20, "\Nearrow\tau"] & \\
a \arrow[u, "f_0"] \arrow[r, "g\circ f_n"'] \arrow[ur, phantom, "\kappa\Uparrow"] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b_0 \arrow[dr, bend left=40, "k"] & \\
a \arrow[u, "f_0"] \arrow[r, "g\circ f_n"'] \arrow[ur, phantom, "\theta\Uparrow"] & c
\end{tikzcd}
$$

$$
\iff\quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] & a \arrow[r, dashed, no head] \arrow[d, "f_0"] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, dashed, no head] \arrow[u, phantom, "\Downarrow Uf_0"'] \arrow[d, "k"'] & b_0 \arrow[r, phantom, "\Downarrow\kappa"] \arrow[d, "l"] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] \arrow[u, phantom, "\Downarrow\tau"'] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "k"'] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

$$
\iff\quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow Uf_0"] & a \arrow[r, dashed, no head] \arrow[d, "f_0"] \arrow[dr, phantom, "\Downarrow\beta_1"] & a \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & a \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & b_0 \arrow[r, "\shortmid" marking, "\vec{J}_1"'] \arrow[d, "l"] & b_1 \arrow[r, "\shortmid" marking, "\vec{J}_2"'] & \cdots \arrow[r, "\shortmid" marking, "\vec{J}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\shortmid" marking, "\vec{J}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & a \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{J}_1"', "\shortmid\shortmid" marking] \arrow[d, "k"'] & b_1 \arrow[r, "\vec{J}_2"', "\shortmid\shortmid" marking] & \cdots \arrow[d, phantom, "\Downarrow\theta'"] \arrow[r, "\vec{J}_{n-1}"', "\shortmid\shortmid" marking] & b_{n-1} \arrow[r, "\vec{J}_n"', "\shortmid\shortmid" marking] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

$$
\iff \quad
\begin{tikzcd}
b_0 \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & b_0 \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
b_0 \arrow[rr, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, "k"'] \arrow[drr, phantom, "\Downarrow\theta'"] & & b_n \arrow[d, "g"] \\
c \arrow[rr, dashed, no head] & & c
\end{tikzcd}
$$

（ここで $\langle\beta_1, \cdots, \beta_n\rangle$ が nullary-弱 cocartesian であることを使った.）

となるから

$$\mathcal{V}(\mathbb{D}) \text{ において } f_0^\dagger(g \circ f_n) = \langle l, \kappa\rangle \text{ である} \iff \eta \text{ が左 Kan 拡張である}$$

である. □

定理 19. $\eta\colon g \Rightarrow l \circ f$ を $\mathcal{V}(\mathbb{D})$ の 2-morphism，即ち下の図式の右辺のような cell とする．f が conjoint を持つとして，これを使って

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, "g"] \\
b \arrow[d, "l"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

と分解する．このとき

$$\mathcal{V}(\mathbb{D}) \text{ において } f^\dagger g = \langle l, \eta\rangle \text{ である} \iff \eta' \text{ が左 Kan 拡張である.}$$

証明. 命題 17 より η_f は cocartesian である．よって補題 18 から分かる. □

$|\vec{J}| = 0$ の場合については次のようになる.

命題 20. $\mathbb{D}$ の次の図式のような cell に対して

$$\eta \text{ が左 Kan 拡張である} \iff \eta \text{ が } \mathcal{V}(\mathbb{D}) \text{ の同型な 2-morphism となる.}$$

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

証明. 定義から容易に分かる. □

左 Kan 拡張の横向きの合成については次の命題が成り立つ.

命題 21. 次の図式において η が左 Kan 拡張のとき

$$\sigma \text{ が左 Kan 拡張である} \iff \text{全体が左 Kan 拡張である.}$$

$$
\begin{tikzcd}
d \arrow[r, "\vec{K}", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

証明. 定義から容易に分かる. □

一方で縦向きの合成については次のようになる.

補題 22. 次の図式において $\langle \beta_1, \cdots, \beta_n \rangle$ は nullary-弱 cocartesian パスであるとする（一番左の垂直射は等号であることに注意する）.

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "\shortmid"{marking}] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortmid"{marking}] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid"{marking}] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[r, "\vec{K}_1"', "\shortmid"{marking}] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"', "\shortmid"{marking}] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid"{marking}] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

このとき

$$\eta \text{ が左 Kan 拡張である} \iff \text{全体が左 Kan 拡張である.}$$

証明. $\langle \beta_1, \cdots, \beta_n \rangle$ が nullary-弱 cocartesian だから

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "\shortmid"{marking}] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortmid"{marking}] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid"{marking}] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[r, "\vec{K}_1"', "\shortmid"{marking}] \arrow[d, "k"'] & b_1 \arrow[r, "\vec{K}_2"', "\shortmid"{marking}] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\theta'"] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid"{marking}] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1{}^\frown\cdots{}^\frown\vec{J}_n", "\shortmid"{marking}] \arrow[dd, "k"'] & a_n \arrow[d, "f_n"] \\
\arrow[r, phantom, "\Downarrow\theta"] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

により θ と θ' が 1 対 1 に対応する．このとき

$$\begin{tikzcd}
a_0 \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & a_0 \arrow[rr, "\vec{K}_1\frown\cdots\frown\vec{K}_n", "\shortmid" marking] \arrow[d, "l"] \arrow[drr, phantom, "\Downarrow\eta"] & & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[rr, dashed, no head] & & c
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "\vec{K}_1\frown\cdots\frown\vec{K}_n", "\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta'"] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}$$

$$\Longleftrightarrow
\begin{tikzcd}
a_0 \arrow[rr, "\vec{J}_1", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\beta_1"] & & a_1 \arrow[r, "\vec{J}_2", "\shortmid" marking] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortmid" marking] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & a_0 \arrow[r, "\vec{K}_1"', "\shortmid" marking] \arrow[d, "l"'] \arrow[drrrr, phantom, "\Downarrow\eta"] & b_1 \arrow[r, "\vec{K}_2"', "\shortmid" marking] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortmid" marking] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid" marking] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1\frown\cdots\frown\vec{J}_n", "\shortmid" marking] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & a_n \arrow[d, "f_n"] \\
 & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}$$

$$\Longleftrightarrow
\begin{tikzcd}
a_0 \arrow[r, dashed, no head] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\tau"] & a_0 \arrow[r, "\vec{J}_1", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "\shortmid" marking] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortmid" marking] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
 & a_0 \arrow[r, "\vec{K}_1"', "\shortmid" marking] \arrow[d, "l"'] \arrow[drrrr, phantom, "\Downarrow\eta"] & b_1 \arrow[r, "\vec{K}_2"', "\shortmid" marking] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortmid" marking] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid" marking] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1\frown\cdots\frown\vec{J}_n", "\shortmid" marking] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & a_n \arrow[d, "f_n"] \\
 & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}$$

となるから

$$\eta \text{ が左 Kan 拡張である} \Longleftrightarrow \text{全体が左 Kan 拡張である}$$

が分かる． $\square$

2-category の場合と同じように次のように定義する．

定義． 次の図式において η が左 Kan 拡張のとき
垂直射 $f\colon a \to b$ が η と**交換する** $\Longleftrightarrow$ 次の図式全体も左 Kan 拡張である．

$$\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & v \arrow[d, "g"] \\
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}$$

定義． 左 Kan 拡張 η が**絶対左 Kan 拡張** $\Longleftrightarrow$ η が任意の垂直射と交換する．

命題 23. $f \dashv u \colon a \to b$ を $\mathcal{V}(\mathbb{D})$ における随伴とするとき，f は任意の左 Kan 拡張と交換する．

証明. $\mathbb{D}$ の cell

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & v \arrow[d, "g"] \\
a \arrow[r, dash, dashed] & a
\end{tikzcd}
$$

を左 Kan 拡張として，$f \dashv u \colon a \to b$ の unit と counit を κ, ε とする．f が η と交換することを示すため

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & v \arrow[d, "g"] \\
 & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

を考える．η が左 Kan 拡張だから次の τ が得られる．

$$
\begin{tikzcd}
u \arrow[r, dash, dashed] \arrow[dd, "k"'] \arrow[dddr, phantom, "\Downarrow\tau"] & u \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[ddd, "l"'] \arrow[dddr, phantom, "\Downarrow\eta"] & v \arrow[ddd, "g"'] \\
 & & \\
b \arrow[d, "u"'] & & \\
a \arrow[r, dash, dashed] & a \arrow[r, dash, dashed] & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & v \arrow[r, dash, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & v \arrow[d, "g"] \\
 & a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[ddr, phantom, "\Downarrow\kappa"] & a \arrow[dd, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow Uu"] & b \arrow[d, "u"] & \\
a \arrow[r, dash, dashed] & a \arrow[r, dash, dashed] & a
\end{tikzcd}
$$

このとき

$$
\begin{tikzcd}
u \arrow[r, dashed] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow Uk"] & u \arrow[r, dashed] \arrow[dd, "k"] \arrow[dddr, phantom, "\Downarrow\tau"] & u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[ddd, "l"] \arrow[dddr, phantom, "\Downarrow\eta"] & v \arrow[ddd, "g"] \\
 & & & \\
b \arrow[r, dashed] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[d, "u"] & & \\
 & a \arrow[r, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed] & b \arrow[r, dashed] & b \arrow[r, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & v \arrow[r, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & v \arrow[d, "g"] \\
 & a \arrow[r, dashed] \arrow[d, "f"'] \arrow[ddr, phantom, "\Downarrow\kappa"] & a \arrow[dd, equal] \\
b \arrow[r, dashed] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[d, "u"] & \\
 & a \arrow[r, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed] & b \arrow[r, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[dddd, "k"'] \arrow[ddddr, phantom, "\Downarrow\theta"] & v \arrow[dd, "g"] \\
 & \\
 & a \arrow[dd, "f"] \\
 & \\
b \arrow[r, dashed] & b
\end{tikzcd}
$$

である．このような cell の一意性を示すため τ' が

$$
\begin{tikzcd}
u \arrow[r, dashed] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\tau'"] & u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & v \arrow[d, "g"] \\
 & a \arrow[r, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed] & b \arrow[r, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & v \arrow[d, "g"] \\
 & a \arrow[d, "f"] \\
b \arrow[r, dashed] & b
\end{tikzcd}
$$

を満たすとする．このとき

$$
\begin{tikzcd}
u \arrow[r, dashed] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\tau'"] & u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & v \arrow[d, "g"] \\
 & a \arrow[r, dashed] \arrow[d, "f"'] \arrow[ddr, phantom, "\Downarrow\kappa"] & a \arrow[dd, equal] \\
b \arrow[r, dashed] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow Uu"] & b \arrow[d, "u"] & \\
a \arrow[r, dashed] & a \arrow[r, dashed] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & v \arrow[r, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & v \arrow[d, "g"] \\
 & a \arrow[r, dashed] \arrow[d, "f"'] \arrow[ddr, phantom, "\Downarrow\kappa"] & a \arrow[dd, equal] \\
b \arrow[r, dashed] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow Uu"] & b \arrow[d, "u"] & \\
a \arrow[r, dashed] & a \arrow[r, dashed] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed] \arrow[dd, "k"'] \arrow[dddr, phantom, "\Downarrow\tau"] & u \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[ddd, "l"] \arrow[dddr, phantom, "\Downarrow\eta"] & v \arrow[ddd, "g"] \\
 & & \\
b \arrow[d, "u"'] & & \\
a \arrow[r, dashed] & a \arrow[r, dashed] & a
\end{tikzcd}
$$

となるから左 Kan 拡張 η の普遍性により

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[dd, "k"'] & u \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[r, phantom, shift right=4ex, "\Downarrow Ul"] & u \arrow[d, "l"] \\
\arrow[r, phantom, "\Downarrow\tau'"] & a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "u"'] & b \arrow[d, "u"] \arrow[r, phantom, "\Downarrow\kappa"] & \\
a \arrow[r, dashed, no head] \arrow[ur, phantom, "\Downarrow Uu"] & a \arrow[r, dashed, no head] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "k"'] & u \arrow[dd, "l"] \\
b \arrow[d, "u"'] \arrow[r, phantom, "\Downarrow\tau"] & \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
$$

となる．よって

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "k"'] & u \arrow[r, dashed, no head] \arrow[d, "k"] & u \arrow[dd, "l"] \\
b \arrow[r, dashed, no head] \arrow[dd, equal] \arrow[ur, phantom, "\Downarrow Uk"] & b \arrow[d, "u"] \arrow[ur, phantom, "\Downarrow\tau"] & \\
\arrow[r, phantom, "\Downarrow\varepsilon"] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] \arrow[ur, phantom, "\Downarrow Uf"] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[dd, "k"'] & u \arrow[r, dashed, no head] \arrow[d, "l"'] & u \arrow[d, "l"] \arrow[l, phantom, shift left=4ex, "\Downarrow Ul"] \\
\arrow[r, phantom, "\Downarrow\tau'"] & a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, equal] \\
b \arrow[r, dashed, no head] \arrow[dd, equal] & b \arrow[d, "u"] \arrow[r, phantom, "\Downarrow\kappa"] & \\
\arrow[r, phantom, "\Downarrow\varepsilon"] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] \arrow[ur, phantom, "\Downarrow Uf"] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[dd, "k"'] & u \arrow[d, "l"] \\
\arrow[r, phantom, "\Downarrow\tau'"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

である． $\square$

命題 24. $f\colon a \to b$ の companion が存在するとき ${}_f\varepsilon$ は絶対左 Kan 拡張である．

証明. 任意の $g\colon b \to c$ に対して

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

が左 Kan 拡張であることを示す．まず companion の定義より

$$
(*)\;
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[dd, "f"'] \arrow[ddr, phantom, "\Downarrow Uf"] & a \arrow[dd, "f"] \\
 & \\
b \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow Ug"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[ddd, "g\circ f"'] \arrow[dddr, phantom, "\Downarrow U(g\circ f)"] & a \arrow[ddd, "g\circ f"] \\
 & \\
 & \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

となり，$U(g \circ f)$ は命題 20 から左 Kan 拡張である．命題 17 より ${}_f\eta$ は cocartesian だから補題 22 により，$(*)$ の部分も左 Kan 拡張である． □

命題 25. $\eta\colon \mathrm{id} \Rightarrow u \circ f$ を $\mathcal{V}(\mathbb{D})$ の 2-morphism とする．f が conjoint を持つとして η を

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow \eta'"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, equal] \\
b \arrow[d, "u"'] \arrow[r, phantom, "\Downarrow \eta"] & \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
$$

と分解する．このとき以下の条件は同値である．

(1) $\mathcal{V}(\mathbb{D})$ において $f \dashv u$ であり，η はその unit である．

(2) η' は cartesian である．

(3) η' は絶対左 Kan 拡張である．

(4) η' は左 Kan 拡張で，f は η' と交換する．

証明. ((1)$\Longrightarrow$(2)) $f \dashv u$ として unit を η，counit を ε とする．η と同様に ε を

$$
\begin{tikzcd}
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon'"] & b \arrow[d, "u"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[r, dashed, no head] \arrow[dd, equal] & b \arrow[d, "u"] \\
\arrow[r, phantom, "\Downarrow \varepsilon"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

と分解する．$\langle f^\star, \varepsilon', \eta' \rangle$ が u の companion を定めることを示せば命題 3 から分かる．

まず

$$
U\mathrm{id} \quad = \quad
\begin{tikzcd}
b \arrow[r, dashed, no head] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[r, dashed, no head] \arrow[d, "u"] \arrow[dr, phantom, "\Downarrow Uu"] & b \arrow[d, "u"] \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[ddr, phantom, "\Downarrow\eta"] & a \arrow[dd, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow Uu"] & b \arrow[d, "u"] & \\
a \arrow[r, dashed, no head] & a \arrow[r, dashed, no head] & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[rr, dashed, no head] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\varepsilon'"] & & b \arrow[d, "u"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "u"'] \arrow[drr, phantom, "\Downarrow\eta'"] & b \arrow[r, "f^\star", "\shortmid"{marking}] & a \arrow[d, equal] \\
a \arrow[rr, dashed, no head] & & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon'"] & b \arrow[d, "u"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
$$

である．同様に

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad U\mathrm{id} \quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[dd, equal] \\
b \arrow[r, dashed, no head] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[d, "u"] \arrow[dr, phantom, "\Downarrow\eta"] & \\
 & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, dashed, no head] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\eta_f"] & & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon'"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "u"] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dashed, no head] & a \arrow[d, "f"] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
$$

となるから η_f が cocartesian，ε_f が cartesian であることより

$$
\begin{tikzcd}
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon'"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "u"] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid"{marking}] & a \arrow[r, dashed, no head] & a
\end{tikzcd}
\quad = \quad \mathrm{id}_{f^\star}
$$

が分かる．

((2)$\Longrightarrow$(3)) η' が cartesian だから，命題 4 により $f^\star$ は u の companion を定める．よって命題 24 により η' は絶対左 Kan 拡張である．

((3)$\Longrightarrow$(4)) 明らか．

((4)$\Longrightarrow$(1)) η' が左 Kan 拡張だから，定理 19 により $\mathcal{V}(\mathbb{D})$ において $f^\dagger\mathrm{id}_a = \langle u, \eta\rangle$ である．f が η と交換するから，f は $f^\dagger\mathrm{id}_a$ とも交換する．故に $\mathcal{V}(\mathbb{D})$ において $f \dashv u$ である． □

4.2 各点 Kan 拡張

pseudo double category の場合と同様に各点左 Kan 拡張を次のように定義する.

定義. $\mathbb{D}$ の cell

$$\begin{tikzcd} a \arrow[r, "\vec{J}"] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \end{tikzcd}$$

が $\vec{J}$ に沿った g の**各点左 Kan 拡張** $\Longleftrightarrow$ 次の右辺のような cell θ に対して，左辺のような τ が一意に存在して

$$\begin{tikzcd} u \arrow[r, "\vec{K}"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & a \arrow[r, "\vec{J}"] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c \end{tikzcd} = \begin{tikzcd} u \arrow[r, "\vec{K}"] \arrow[d, "k"'] & a \arrow[r, "\vec{J}"] \arrow[d, phantom, "\Downarrow\theta"] & b \arrow[d, "g"] \\ c \arrow[rr, dashed, no head] & & c \end{tikzcd}$$

となる.

定義. 次の図式において η が各点左 Kan 拡張のとき
垂直射 $f\colon a \to b$ が η と**交換する** $\Longleftrightarrow$ 次の図式全体も各点左 Kan 拡張である.

$$\begin{tikzcd} u \arrow[r, "\vec{J}"] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & v \arrow[d, "g"] \\ a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

定義. 各点左 Kan 拡張 η が**絶対各点左 Kan 拡張** $\Longleftrightarrow$ η が任意の垂直射と交換する.

定義から明らかに各点左 Kan 拡張は左 Kan 拡張である.
左 Kan 拡張と同様にして以下の命題が成り立つ.

命題 26. 次の図式において η が各点左 Kan 拡張のとき

σ が各点左 Kan 拡張である $\Longleftrightarrow$ 全体が各点左 Kan 拡張である.

$$\begin{tikzcd}
d \arrow[r, "\vec{K}"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, "\vec{J}"] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}$$

□

命題 27. $\mathcal{V}(\mathbb{D})$ における左随伴は任意の各点左 Kan 拡張と交換する. □

命題 28. ${}_f\varepsilon$ は絶対各点左 Kan 拡張である. □

各点左 Kan 拡張に対して補題 22 と同様のことを考える場合は nullary-左 cocartesian を使うことになる.

補題 29. 次の図式において $\langle\beta_1, \cdots, \beta_n\rangle$ は nullary-左 cocartesian パスであるとする.

$$\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[r, "\vec{K}_1"'] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}$$

このとき

$$\eta \text{ が各点左 Kan 拡張である} \iff \text{全体が各点左 Kan 拡張である.}$$

証明. 補題 22 と同様にして

$$\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & a_0 \arrow[r, "\vec{K}_1{}^\frown\cdots{}^\frown\vec{K}_n"] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}{}^\frown\vec{K}_1{}^\frown\cdots{}^\frown\vec{K}_n"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta'"] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}$$

$$\iff
\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a_0 \arrow[r, "\vec{J}_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
u \arrow[r, "\vec{H}"'] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & a_0 \arrow[r, "\vec{K}_1"'] \arrow[d, "l"] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}{}^\frown\vec{J}_1{}^\frown\cdots{}^\frown\vec{J}_n"] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & a_n \arrow[d, "f_n"] \\
 & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}$$

$$
\iff
\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\tau"] & a_0 \arrow[r, "\vec{J}_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
{} & a_0 \arrow[r, "\vec{K}_1"'] \arrow[d, "l"'] \arrow[drrrr, phantom, "\Downarrow\eta"] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}^\frown\vec{J}_1^\frown\cdots^\frown\vec{J}_n"] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\theta"] & a_n \arrow[d, "f_n"] \\
& b_n \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

となるから分かる. □

4.3 r-各点 Kan 拡張

定義. $n \geq 1$ のとき，左 Kan 拡張

$$
\begin{tikzcd}
a_0 \arrow[r, "J_1"] \arrow[d, "l"'] \arrow[drrrr, phantom, "\Downarrow\eta"] & a_1 \arrow[r, "J_2"] & \cdots \arrow[r, "J_{n-1}"] & a_{n-1} \arrow[r, "J_n"] & a_n \arrow[d, "g"] \\
b \arrow[rrrr, dashed, no head] & & & & b
\end{tikzcd}
$$

が **r-各点左 Kan 拡張**

$\iff k\colon u \to a_0$ として制限 $\langle J_1(k, \mathrm{id}), \beta\rangle$ が存在するとき，合成

$$
\begin{tikzcd}
u \arrow[r, "J_1(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta"] & a_1 \arrow[r, "J_2"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a_2 \arrow[r, "J_3"] \arrow[d, equal] & \cdots \arrow[r, "J_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "J_n"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a_n \arrow[d, equal] \\
a_0 \arrow[r, "J_1"'] \arrow[d, "l"'] \arrow[drrrrr, phantom, "\Downarrow\eta"] & a_1 \arrow[r, "J_2"'] & a_2 \arrow[r, "J_3"'] & \cdots \arrow[r, "J_{n-1}"'] & a_{n-1} \arrow[r, "J_n"'] & a_n \arrow[d, "g"] \\
b \arrow[rrrrr, dashed, no head] & & & & & b
\end{tikzcd}
\tag{30}
$$

が左 Kan 拡張になる.

補題 31. η が r-各点左 Kan 拡張ならば図式 (30) も r-各点左 Kan 拡張である.

証明. cartesian cell の合成が cartesian であること（補題 5）より明らか. □

命題 32. 次の図式において η が左 Kan 拡張で $|\vec{K}| \geq 1$ のとき

$$\sigma \text{ が r-各点左 Kan 拡張である} \iff \text{全体が r-各点左 Kan 拡張である.}$$

$$
\begin{tikzcd}
d \arrow[r, "\vec{K}", "\shortmid\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

証明. 定義と命題 21 から容易に分かる. □

r-各点左 Kan 拡張に対する補題 22 を述べるため，次の定義をする.

定義. 次のような $\langle\beta_1, \cdots, \beta_n\rangle$ が nullary-弱 cocartesian パスであるとする.

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortmid\shortmid" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[d, "f_1"] \\
b_0 \arrow[r, "\vec{K}_1"', "\shortmid\shortmid" marking] & b_1
\end{tikzcd}
, \quad \cdots, \quad
\begin{tikzcd}
a_{n-1} \arrow[r, "\vec{J}_n", "\shortmid\shortmid" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_n \arrow[d, "f_n"] \\
b_{n-1} \arrow[r, "\vec{K}_n"', "\shortmid\shortmid" marking] & b_n
\end{tikzcd}
$$

このとき $\langle\beta_1, \cdots, \beta_n\rangle$ が **nullary-r-左弱 cocartesian パス** $\Longleftrightarrow$ 次の条件を満たす.

(1) $f_0 = \mathrm{id}_{a_0}$ である（従って $a_0 = b_0$ である）.
(2) $|\vec{J}_1| \geq 1$ である（$\vec{J}_1 = \langle a_0, J_{11}, x, \cdots, z, J_{1m}, a_1\rangle$ と書く）.
(3) $|\vec{K}_1| = 1$ である（$\vec{K}_1 = \langle a_0, K_1, b_1\rangle$ と書く）.
(4) $k\colon u \to a_0$ が垂直射で制限 $\langle J_{11}(k, \mathrm{id}), \gamma\rangle$ が存在するとき，制限 $\langle K_1(k, \mathrm{id}), \sigma\rangle$ も存在して，更に

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}'_1", "\shortmid\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta'_1"] & a_1 \arrow[d, "f_1"] \\
u \arrow[r, "K_1(k,\mathrm{id})", "\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b_1 \arrow[d, equal] \\
a_0 \arrow[r, "K_1"', "\shortmid" marking] & b_1
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "J_{11}(k,\mathrm{id})", "\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[r, "J_{12}", "\shortmid" marking] \arrow[d, equal] & \cdots \arrow[r, "J_{1m}", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\mathrm{id}"] & a_1 \arrow[d, equal] \\
a_0 \arrow[r, "J_{11}"', "\shortmid" marking] \arrow[d, equal] & x \arrow[r, "J_{12}"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\beta_1"] & \cdots \arrow[r, "J_{1m}"', "\shortmid" marking] & a_1 \arrow[d, "f_1"] \\
a_0 \arrow[rrr, "K_1"', "\shortmid" marking] & & & b_1
\end{tikzcd}
$$

により β'_1 を定義すると $\langle\beta'_1, \beta_2, \cdots, \beta_n\rangle$ が nullary-弱 cocartesian パスである.

定義中の nullary-弱 cocartesian を全て nullary-左 cocartesian に変えることで **nullary-r-左 cocartesian パス**も定義する[*8].

[*8] unary-r-左弱 cocartesian や r-左弱 cocartesian なども定義できるが，以下では使用しないためここでは定義しない.

補題 33. 次の図式において $\langle \beta_1, \cdots, \beta_n \rangle$ は nullary-r-左弱 cocartesian パスであるとする.

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[r, "K_1"'] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed] & & & & c
\end{tikzcd}
$$

このとき

$$\eta \text{ が r-各点左 Kan 拡張である} \Longrightarrow \text{全体が r-各点左 Kan 拡張である.}$$

任意の $k\colon u \to a_0$ に対して制限 $J_{11}(k, \mathrm{id})$ が存在するならば逆も成り立つ.

証明. ($\Longrightarrow$) η が r-各点左 Kan 拡張であるとする. 全体が r-各点左 Kan 拡張であることを示すため $k\colon u \to a_0$ を取り制限 $\langle J_{11}(k, \mathrm{id}), \gamma \rangle$ が存在するとする. このとき nullary-r-左弱 cocartesian パスの定義より制限 $\langle K_1(k, \mathrm{id}), \sigma \rangle$ も存在して

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}'_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta'_1"] & a_1 \arrow[d, "f_1"] \\
u \arrow[r, "K_1(k{,}\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b_1 \arrow[d, equal] \\
a_0 \arrow[r, "K_1"'] & b_1
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "J_{11}(k{,}\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[r, "J_{12}"] \arrow[d, equal] & \cdots \arrow[r, "J_{1m}"] \arrow[d, phantom, "\Downarrow\mathrm{id}"] & a_1 \arrow[d, equal] \\
a_0 \arrow[r, "J_{11}"'] \arrow[d, equal] & x \arrow[r, "J_{12}"'] & \cdots \arrow[r, "J_{1m}"'] \arrow[d, phantom, "\Downarrow\beta_1"] & a_1 \arrow[d, "f_1"] \\
a_0 \arrow[rrr, "K_1"'] & & & b_1
\end{tikzcd}
$$

により β'_1 を定義すると $\langle \beta'_1, \beta_2, \cdots, \beta_n \rangle$ が nullary-弱 cocartesian パスである. η が r-各点左 Kan 拡張だから

$$
\begin{tikzcd}
u \arrow[r, "K_1(k{,}\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b_1 \arrow[r, "\vec{K}_2"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b_2 \arrow[r, "\vec{K}_3"] \arrow[d, equal] & \cdots \arrow[r, "\vec{K}_{n-1}"] \arrow[d, phantom, "\cdots"] & b_{n-1} \arrow[r, "\vec{K}_n"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b_n \arrow[d, equal] \\
a_0 \arrow[r, "K_1"'] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"'] & b_2 \arrow[r, "\vec{K}_3"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrrr, dashed] & & & & & c
\end{tikzcd}
$$

は左 Kan 拡張である．よって補題 22 を使えば

$$
\begin{tikzcd}
u \arrow[r, "J_{11}(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[r, "J_{12}"] \arrow[d, equal] & \cdots \arrow[r, "J_{1m}"] \arrow[d, phantom, "\Downarrow\mathrm{id}"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, equal] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a_n \arrow[d, equal] \\
a_0 \arrow[r, "J_{11}"'] \arrow[d, equal] & x \arrow[r, "J_{12}"'] \arrow[d, phantom, "\Downarrow\beta_1"] & \cdots \arrow[r, "J_{1m}"'] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"'] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[rrr, "K_1"'] \arrow[d, "l"'] & & & b_1 \arrow[r, "\vec{K}_2"'] \arrow[d, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "\vec{K}_{n-1}"'] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrrrr, dashed] & & & & & & c
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
u \arrow[r, "\vec{J}_1'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1'"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\Downarrow\beta_2"] & a_2 \arrow[r, "\vec{J}_3"] \arrow[d, "f_2"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
u \arrow[r, "K_1(k,\mathrm{id})"'] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b_1 \arrow[r, "\vec{K}_2"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b_2 \arrow[r, "\vec{K}_3"'] \arrow[d, equal] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\cdots"] & b_{n-1} \arrow[r, "\vec{K}_n"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b_n \arrow[d, equal] \\
a_0 \arrow[r, "K_1"'] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"'] & b_2 \arrow[r, "\vec{K}_3"'] \arrow[d, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "\vec{K}_{n-1}"'] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrrr, dashed] & & & & & c
\end{tikzcd}
$$

も左 Kan 拡張である．

($\Longleftarrow$) 同様. $\square$

定義. 水平射 $J\colon a \nrightarrow b$ の **tabulation** とは組 $\langle\langle J\rangle, \pi\rangle$ であって

(1) $\langle J\rangle \in \mathbb{D}$ は対象で π は次のような cell である．

$$
\begin{tikzcd}
\langle J\rangle \arrow[r, dashed] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J\rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"'] & b
\end{tikzcd}
$$

(2)（1 次元的普遍性）次の右辺のような cell φ に対して，垂直射 $h\colon u \to \langle J\rangle$ が一意に

存在して

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle J \rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow \pi"] & \langle J \rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[dd, "q_a"'] \arrow[ddr, phantom, "\Downarrow \varphi"] & u \arrow[dd, "q_b"] \\
 & \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
$$

となる．

(3)（2 次元的普遍性）2 つの cell

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow \varphi"] & u \arrow[d, "q_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
v \arrow[r, dashed, no head] \arrow[d, "r_a"'] \arrow[dr, phantom, "\Downarrow \psi"] & v \arrow[d, "r_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
$$

に対して，1 次元的普遍性により

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle J \rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow \pi"] & \langle J \rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[dd, "q_a"'] \arrow[ddr, phantom, "\Downarrow \varphi"] & u \arrow[dd, "q_b"] \\
 & \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
\quad
\begin{tikzcd}
v \arrow[r, dashed, no head] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & v \arrow[d, "k"] \\
\langle J \rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow \pi"] & \langle J \rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
=
\begin{tikzcd}
v \arrow[r, dashed, no head] \arrow[dd, "r_a"'] \arrow[ddr, phantom, "\Downarrow \psi"] & v \arrow[dd, "r_b"] \\
 & \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
$$

となるように h, k を取る．更に $\vec{H}$ と cell ξ_a, ξ_b が等式

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow \varphi"] & u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "q_b"] \arrow[dr, phantom, "\Downarrow \xi_b"] & v \arrow[d, "r_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow \xi_a"] & v \arrow[r, dashed, no head] \arrow[d, "r_a"] \arrow[dr, phantom, "\Downarrow \psi"] & v \arrow[d, "r_b"] \\
a \arrow[r, dashed, no head] & a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
$$

を満たすとする．このとき cell

$$
\begin{tikzcd}
u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow \xi"] & v \arrow[d, "k"] \\
\langle J \rangle \arrow[r, dashed, no head] & \langle J \rangle
\end{tikzcd}
$$

が一意に存在して等式

$$
\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[d, "h"'] & v \arrow[d, "k"] \\
\langle J\rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[u, phantom, "\Downarrow\xi"] & \langle J\rangle \arrow[d, "p_a"] \\
a \arrow[r, dashed, no head] \arrow[u, phantom, "\Downarrow Up_a"] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[dd, "q_a"'] & v \arrow[dd, "r_a"] \\
\arrow[r, phantom, "\Downarrow\xi_a"] & {} \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[d, "h"'] & v \arrow[d, "k"] \\
\langle J\rangle \arrow[r, dashed, no head] \arrow[d, "p_b"'] \arrow[u, phantom, "\Downarrow\xi"] & \langle J\rangle \arrow[d, "p_b"] \\
b \arrow[r, dashed, no head] \arrow[u, phantom, "\Downarrow Up_b"] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}"] \arrow[dd, "q_b"'] & v \arrow[dd, "r_b"] \\
\arrow[r, phantom, "\Downarrow\xi_b"] & {} \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

が成り立つ.

更に π が nullary-弱 cocartesian のとき，$\langle\langle J\rangle, \pi\rangle$ を **nullary-弱 cocartesian tabulation** という.

命題 34. $f\colon a \to x$，$g\colon b \to x$ を $\mathbb{D}$ の垂直射として，制限 $\langle x(f,g), \sigma\rangle$ が存在して更にその tabulation $\langle\langle x(f,g)\rangle, \pi\rangle$ が存在するとする.

$$
\begin{tikzcd}
\langle x(f,g)\rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
$$

このとき $\mathcal{V}(\mathbb{D})$ において $\langle x(f,g)\rangle$ はコンマ対象 $g{\downarrow}f$ となる.

証明. 上記の cell の合成 $\kappa := \sigma \circ \pi$ は $\mathcal{V}(\mathbb{D})$ における次の図式を与える.

$$
\begin{tikzcd}
a \arrow[r, "f"] & x \\
\langle x(f,g)\rangle \arrow[u, "p_a"] \arrow[r, "p_b"'] \arrow[ur, phantom, "\Uparrow\kappa"] & b \arrow[u, "g"']
\end{tikzcd}
$$

これがコンマ対象を与えることを示そう.

（1 次元的普遍性）$\mathcal{V}(\mathbb{D})$ の任意の図式

$$
\begin{tikzcd}
a \arrow[r, "f"] & x \\
u \arrow[u, "q_a"] \arrow[r, "q_b"'] \arrow[ur, phantom, "\Uparrow\varphi"] & b \arrow[u, "g"']
\end{tikzcd}
$$

を取る．これを $\mathbb{D}$ における cell とみなして

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & u \arrow[d, "q_b"] \\
a \arrow[r, "x(f{,}g)", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "q_a"'] & u \arrow[d, "q_b"] \\
a \arrow[r, phantom, "\Downarrow\varphi"] \arrow[d, "f"'] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
$$

と分解する．この φ' に対して tabulation の 1 次元的普遍性により $h\colon u \to \langle x(f,g) \rangle$ が一意に存在して

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle x(f,g) \rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle x(f,g) \rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f{,}g)"', "|" marking] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & u \arrow[d, "q_b"] \\
a \arrow[r, "x(f{,}g)"', "|" marking] & b
\end{tikzcd}
$$

となる．このとき

$$
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[dr, phantom, "\Uparrow\kappa"] & x \\
 & \langle x(f,g) \rangle \arrow[u, "p_a"] \arrow[r, "p_b"'] & b \arrow[u, "g"'] \\
u \arrow[ur, "h"] \arrow[uur, bend left, "q_a"] \arrow[urr, bend right, "q_b"'] \arrow[u, phantom, "="] \arrow[urr, phantom, "="] & &
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[d, phantom, "\Uparrow\varphi"] & x \\
 & & b \arrow[u, "g"'] \\
u \arrow[uur, bend left, "q_a"] \arrow[urr, bend right, "q_b"'] & &
\end{tikzcd}
$$

である．h の一意性も分かる．

（2 次元的普遍性）$h, k\colon u \to \langle x(f,g) \rangle$ を $\mathcal{V}(\mathbb{D})$ の 1-morphism として，2-morphism ξ_a, ξ_b が

$$
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[dr, phantom, "\Uparrow\kappa"] & x \\
 & \langle x(f,g) \rangle \arrow[u, "p_a"] \arrow[r, "p_b"'] & b \arrow[u, "g"'] \\
u \arrow[ur, "h"] \arrow[uur, bend left, "p_a \circ k"] \arrow[urr, bend right, "p_b \circ h"'] \arrow[u, phantom, "\Uparrow\xi_a"] \arrow[urr, phantom, "="] & &
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[dr, phantom, "\Uparrow\kappa"] & x \\
 & \langle x(f,g) \rangle \arrow[u, "p_a"] \arrow[r, "p_b"'] & b \arrow[u, "g"'] \\
u \arrow[ur, "k"] \arrow[uur, bend left, "p_a \circ k"] \arrow[urr, bend right, "p_b \circ h"'] \arrow[u, phantom, "="] \arrow[urr, phantom, "\Uparrow\xi_b"] & &
\end{tikzcd}
$$

を満たすとする．これは $\mathbb{D}$ の図式で描くと

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[r, dashed, no head] \arrow[d, "h"] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\kappa\bullet h"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \arrow[r, phantom, "\Downarrow\xi_b"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[d, "f"'] & b \arrow[r, dashed, no head] \arrow[d, "g"] \arrow[r, phantom, "\Downarrow Ug"] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x \arrow[r, dashed, no head] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[r, dashed, no head] \arrow[d, "k"] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\xi_a"] & \langle x(f,g)\rangle \arrow[d, "p_a"] \arrow[r, phantom, "\Downarrow\kappa\bullet k"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x \arrow[r, dashed, no head] & x
\end{tikzcd}
$$

$\kappa \bullet h$ と $\kappa \bullet k$ を

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[d, "h"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\varphi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow\sigma"'] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[d, "h"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\kappa\bullet h"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[d, "f"'] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
$$

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "k"'] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\psi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow\sigma"'] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "k"'] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\kappa\bullet k"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[d, "f"'] & b \arrow[d, "g"] \\
x \arrow[r, dashed, no head] & x
\end{tikzcd}
$$

と分解すれば，σ が cartesian だから

$$
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[r, dashed, no head] \arrow[d, "h"] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\varphi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \arrow[r, phantom, "\Downarrow\xi_b"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)"', "\shortmid"{marking}] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[r, dashed, no head] \arrow[d, "k"] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\xi_a"] & \langle x(f,g)\rangle \arrow[d, "p_a"] \arrow[r, phantom, "\Downarrow\psi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, dashed, no head] & a \arrow[r, "x(f,g)"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

が得られる．故に tabulation の 2 次元的普遍性により，cell

$$\begin{tikzcd} u \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & u \arrow[d, "k"] \\ \langle x(f,g)\rangle \arrow[r, dashed, no head] & \langle x(f,g)\rangle \end{tikzcd}$$

が一意に存在して等式

$$\begin{tikzcd} u \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & u \arrow[d, "k"] \\ \langle x(f,g)\rangle \arrow[r, dashed, no head] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow Up_a"] & \langle x(f,g)\rangle \arrow[d, "p_a"] \\ a \arrow[r, dashed, no head] & a \end{tikzcd} = \begin{tikzcd} u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[d, "k"] \\ \langle x(f,g)\rangle \arrow[d, "p_a"'] \arrow[r, phantom, "\Downarrow\xi_a"] & \langle x(f,g)\rangle \arrow[d, "p_a"] \\ a \arrow[r, dashed, no head] & a \end{tikzcd}$$

$$\begin{tikzcd} u \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & u \arrow[d, "k"] \\ \langle x(f,g)\rangle \arrow[r, dashed, no head] \arrow[d, "p_b"'] \arrow[dr, phantom, "\Downarrow Up_b"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd} = \begin{tikzcd} u \arrow[r, dashed, no head] \arrow[d, "h"'] & u \arrow[d, "k"] \\ \langle x(f,g)\rangle \arrow[d, "p_b"'] \arrow[r, phantom, "\Downarrow\xi_b"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が成り立つ．これは $\mathcal{V}(\mathbb{D})$ において

$$\begin{tikzcd} & a \\ & \langle x(f,g)\rangle \arrow[u, "p_a"] \\ u \arrow[ur, bend left, "k"] \arrow[ur, bend right, "h"'] \arrow[ur, phantom, "\Uparrow\xi"] \end{tikzcd} = \begin{tikzcd} & a \\ & \langle x(f,g)\rangle \arrow[u, "p_a"'] \\ u \arrow[uur, bend left, "p_a\circ k"] \arrow[ur, bend right, "h"'] \arrow[ur, phantom, "\Uparrow\xi_a"] \end{tikzcd} \qquad \begin{tikzcd} & \langle x(f,g)\rangle \arrow[r, "p_b"'] & b \\ u \arrow[ur, bend left, "k"] \arrow[ur, bend right, "h"'] \arrow[ur, phantom, "\Uparrow\xi"] \end{tikzcd} = \begin{tikzcd} & \langle x(f,g)\rangle \arrow[r, "p_b"'] & b \\ u \arrow[ur, bend left, "k"] \arrow[urr, bend right, "p_b\circ h"'] \arrow[ur, phantom, "\Uparrow\xi_b"] \end{tikzcd}$$

を意味する． □

命題 35. $\mathbb{D}$ は左制限と nullary-弱 cocartesian tabulation を持つとする．$\eta\colon g \Rightarrow l \circ f$ を $\mathcal{V}(\mathbb{D})$ の 2-morphism，即ち次の右辺のような cell とする．f が conjoint を持つとして，これを使って

$$\begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\ b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \end{tikzcd} = \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, "g"] \\ b \arrow[d, "l"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \\ c \arrow[r, dashed, no head] & c \end{tikzcd}$$

と分解する．このとき

$$\mathcal{V}(\mathbb{D}) \text{ において } f^{\dagger}g = \langle l, \eta \rangle \text{ が c-各点左 Kan 拡張} \iff \eta' \text{ が r-各点左 Kan 拡張.}$$

証明. まず $\mathbb{D}$ が左制限を持つから，垂直射 $j\colon x \to b$ に対して cartesian cell

$$\begin{tikzcd}
x \arrow[r, "f^{\star}(j,\mathrm{id})", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
b \arrow[r, "f^{\star}"', "|" marking] & a
\end{tikzcd}$$

が存在する．これを使って

$$\gamma \quad := \quad \begin{tikzcd}
x \arrow[r, "f^{\star}(j,\mathrm{id})", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
b \arrow[r, "f^{\star}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, equal] & b
\end{tikzcd}$$

と定めると，β も ε_f も cartesian だから γ も cartesian である．故に γ は制限 $b(j, f)$ を定める．よって $f^{\star}(j, \mathrm{id})$ の nullary-弱 cocartesian tabulation

$$\begin{tikzcd}
\langle f^{\star}(j,\mathrm{id}) \rangle \arrow[r, dashed, no head] \arrow[d, "p_x"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle f^{\star}(j,\mathrm{id}) \rangle \arrow[d, "p_a"] \\
x \arrow[r, "f^{\star}(j,\mathrm{id})"', "|" marking] & a
\end{tikzcd}$$

を取れば命題 34 により $\langle f^{\star}(j, \mathrm{id}) \rangle$ は $f \downarrow j$ である．ここで定義より

$$\begin{tikzcd}
x \arrow[r, "f^{\star}(j,\mathrm{id})", "|" marking] \arrow[dd, "j"'] \arrow[ddr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[dd, "f"] \arrow[dddr, phantom, "\Downarrow\eta"] & a \arrow[dddd, "g"] \\
& & \\
b \arrow[r, dashed, no head] \arrow[dd, "l"'] \arrow[ddr, phantom, "\Downarrow Ul"] & b \arrow[dd, "l"] & \\
& & \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "f^{\star}(j,\mathrm{id})", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
b \arrow[r, "f^{\star}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[r, "f^{\star}", "|" marking] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "f^{\star}(j,\mathrm{id})", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
b \arrow[r, "f^{\star}", "|" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\tag{36}$$

となる．これらを使うと

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "|" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\text{ が r-各点左 Kan 拡張である}
$$

$$
\Longleftrightarrow \text{任意の } j \text{ に対して }
\begin{tikzcd}
x \arrow[r, "f^\star(j,\mathrm{id})", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "|" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\text{ が左 Kan 拡張である（定義）}
$$

$$
\Longleftrightarrow \text{任意の } j \text{ に対して }
\begin{tikzcd}
x \arrow[r, "f^\star(\mathrm{id},j)", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[dd, "g"] \\
b \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \arrow[r, phantom, "\Downarrow\eta"] & {} \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
\text{ が左 Kan 拡張である (36)}
$$

$$
\Longleftrightarrow \text{任意の } j \text{ に対して }
\begin{tikzcd}
x \arrow[r, "j"] & b \arrow[dr, "l"] & \\
\langle f^\star(j,\mathrm{id})\rangle \arrow[u, "p_x"] \arrow[r, "p_a"'] \arrow[ur, phantom, "\Nwarrow"] & a \arrow[u, "f"'] \arrow[r, "g"'] \arrow[ur, phantom, "\eta\Uparrow"] & c
\end{tikzcd}
\text{ が左 Kan 拡張 である}
$$

（π が nullary-弱 cocartesian なので，次の図式に補題 18 を使う）

$$
\begin{tikzcd}
\langle f^\star(j,\mathrm{id})\rangle \arrow[rr, dashed, no head] \arrow[d, "p_x"'] & \arrow[d, phantom, "\Downarrow\pi"] & \langle f^\star(j,\mathrm{id})\rangle \arrow[d, "p_a"] \\
x \arrow[r, "f^\star(j,\mathrm{id})", "|" marking] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[dd, "g"] \\
b \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \arrow[r, phantom, "\Downarrow\eta"] & {} \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

□

命題 37. $\mathbb{D}$ の cell

$$
\begin{tikzcd}
a_0 \arrow[r, "J_1", "|" marking] \arrow[d, "l"'] & \cdots \arrow[r, "J_n", "|" marking] \arrow[d, phantom, "\Downarrow\eta"] & a_n \arrow[d, "g"] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
$$

を r-各点左 Kan 拡張として，$f\colon b \to c$ は任意の左 Kan 拡張と交換するとする．r-各点左

Kan 拡張は左 Kan 拡張だから合成

$$\begin{tikzcd}
a_0 \arrow[r, "J_1", "\shortmid" marking] \arrow[d, "l"'] \arrow[drr, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "J_n", "\shortmid" marking] & a_n \arrow[d, "g"] \\
b \arrow[rr, dashed, no head] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow Uf"] & & b \arrow[d, "f"] \\
c \arrow[rr, dashed, no head] & & c
\end{tikzcd}$$

は左 Kan 拡張となるが，これは r-各点左 Kan 拡張である．

証明. r-各点左 Kan 拡張であることを示すため $k\colon u \to a_0$ として制限 $\langle J_1(k,\mathrm{id}), \beta\rangle$ が存在するとする．合成

$$\begin{tikzcd}
u \arrow[r, "J_1(k{,}\mathrm{id})", "\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta"] & a_1 \arrow[r, "J_2", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mathrm{id}"] & \cdots \arrow[r, "J_n", "\shortmid" marking] & a_n \arrow[d, equal] & \\
a_0 \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "l"'] \arrow[drrr, phantom, "\Downarrow\eta"] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_n"', "\shortmid" marking] & a_n \arrow[d, "g"] & (*) \\
b \arrow[rrr, dashed, no head] \arrow[d, "f"'] \arrow[drrr, phantom, "\Downarrow Uf"] & & & b \arrow[d, "f"] & \\
c \arrow[rrr, dashed, no head] & & & c &
\end{tikzcd}$$

が左 Kan 拡張であることを示せばよい．まず η が r-各点左 Kan 拡張だから $(*)$ の部分は左 Kan 拡張である．よって f が任意の左 Kan 拡張と交換するから全体も左 Kan 拡張である． □

4.4　r⁺-各点 Kan 拡張

定義. $n \geq 1$ のとき，左 Kan 拡張

$$\begin{tikzcd}
a_0 \arrow[r, "J_1", "\shortmid" marking] \arrow[d, "l"'] \arrow[drrrr, phantom, "\Downarrow\eta"] & a_1 \arrow[r, "J_2", "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}", "\shortmid" marking] & a_{n-1} \arrow[r, "J_n", "\shortmid" marking] & a_n \arrow[d, "g"] \\
b \arrow[rrrr, dashed, no head] & & & & b
\end{tikzcd}$$

が **r$^+$-各点左 Kan 拡張**

$\Longleftrightarrow$ η が各点左 Kan 拡張であり，更に $k\colon u \to a_0$ に対して制限 $\langle J_1(k,\mathrm{id}), \beta\rangle$ が存在す

るならば，合成

$$
\begin{tikzcd}
u \arrow[r, "J_1(k,\mathrm{id})", "|" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta"] & a_1 \arrow[r, "J_2", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a_2 \arrow[r, "J_3", "|" marking] \arrow[d, equal] & \cdots \arrow[r, "J_{n-1}", "|" marking] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "J_n", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a_n \arrow[d, equal] \\
a_0 \arrow[r, "J_1"', "|" marking] \arrow[d, "l"'] & a_1 \arrow[r, "J_2"', "|" marking] & a_2 \arrow[r, "J_3"', "|" marking] & \cdots \arrow[r, "J_{n-1}"', "|" marking] \arrow[d, phantom, "\Downarrow\eta"] & a_{n-1} \arrow[r, "J_n"', "|" marking] & a_n \arrow[d, "g"] \\
b \arrow[rrrrr, dashed] & & & & & b
\end{tikzcd}
$$

が各点左 Kan 拡張になる．

r-各点左 Kan 拡張のときと同様に次の命題が成り立つ．

命題 38. 次の図式において η が r$^+$-各点左 Kan 拡張のとき

$$\sigma \text{ が r}^+\text{-各点左 Kan 拡張である} \iff \text{全体が r}^+\text{-各点左 Kan 拡張である．}$$

$$
\begin{tikzcd}
d \arrow[r, "\vec{K}"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[r, "\vec{J}"] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "g"] \\
c \arrow[r, dashed] & c \arrow[r, dashed] & c
\end{tikzcd}
$$

□

補題 39. 次の図式において $\langle \beta_1, \cdots, \beta_n \rangle$ は nullary-r-左 cocartesian パスであるとする．

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2"] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}"] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "\vec{J}_n"] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
a_0 \arrow[r, "K_1"', "|" marking] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"'] & \cdots \arrow[r, "\vec{K}_{n-1}"'] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"'] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed] & & & & c
\end{tikzcd}
$$

このとき

$$\eta \text{ が r}^+\text{-各点左 Kan 拡張である} \implies \text{全体が r}^+\text{-各点左 Kan 拡張である．}$$

任意の $k \colon u \to a_0$ に対して制限 $J_{11}(k, \mathrm{id})$ が存在するならば逆も成り立つ． □

命題 40. $\mathbb{D}$ が左制限と companion を持つとき各点左 Kan 拡張は r$^+$-各点左 Kan 拡張である．

証明. 次の図式の η が各点左 Kan 拡張だとして $k \colon u \to a_0$ を取る．まず命題 28 より ${}_k\varepsilon$

が絶対各点左 Kan 拡張だから，命題 26 により

$$
\begin{tikzcd}
u \arrow[r, "k_\star", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow k\varepsilon"] & a_0 \arrow[r, "J_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{J_1}"] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "J_{n-1}", "\shortmid"{marking}] \arrow[d, phantom, "\cdots"] & a_{n-1} \arrow[r, "J_n", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{J_n}"] & a_n \arrow[d, equal] \\
a_0 \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & a_0 \arrow[r, "J_1"', "\shortmid"{marking}] \arrow[d, "l"] & a_1 \arrow[r, "J_2"', "\shortmid"{marking}] & \cdots \arrow[r, "J_{n-1}"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \eta"] & a_{n-1} \arrow[r, "J_n"', "\shortmid"{marking}] & a_n \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[rrrr, dashed, no head] & & & & b
\end{tikzcd}
$$

も各点左 Kan 拡張である．次に左制限 $\langle J_1(k, \mathrm{id}), \beta\rangle$ が存在するから補題 12 より co-cartesian cell

$$
\begin{tikzcd}
u \arrow[r, "k_\star", "\shortmid"{marking}] \arrow[d, equal] & a_0 \arrow[r, "J_1", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\varphi"] & a_1 \arrow[d, equal] \\
u \arrow[rr, "J_1(k,\mathrm{id})"', "\shortmid"{marking}] & & a_1
\end{tikzcd}
$$

が存在する．これは補題 12 の証明によれば

$$
\begin{tikzcd}
u \arrow[r, "k_\star", "\shortmid"{marking}] \arrow[d, equal] & a_0 \arrow[r, "J_1", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\varphi"] & a_1 \arrow[d, equal] \\
u \arrow[rr, "J_1(k,\mathrm{id})"', "\shortmid"{marking}] \arrow[d, "k"'] & \arrow[d, phantom, "\Downarrow\beta"] & a_1 \arrow[d, equal] \\
a_0 \arrow[rr, "J_1"', "\shortmid"{marking}] & & a_1
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "k_\star", "\shortmid"{marking}] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow k\varepsilon"] & a_0 \arrow[r, "J_1", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow \mathrm{id}_{J_1}"] & a_1 \arrow[dd, equal] \\
& & \\
a_0 \arrow[r, dashed, no head] & a_0 \arrow[r, "J_1"', "\shortmid"{marking}] & a_1
\end{tikzcd}
$$

を満たす．従って

$$
\begin{tikzcd}
u \arrow[r, "k_\star", "\shortmid"{marking}] \arrow[d, equal] & a_0 \arrow[r, "J_1(k,\mathrm{id})", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\varphi"] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "J_n", "\shortmid"{marking}] & a_n \arrow[dd, equal] \\
u \arrow[rr, "J_1"', "\shortmid"{marking}] \arrow[d, "k"'] & \arrow[d, phantom, "\Downarrow\beta"] & a_1 \arrow[d, equal] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & & \\
a_0 \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & a_0 \arrow[r, "J_1"', "\shortmid"{marking}] \arrow[d, "l"] & a_1 \arrow[r, "J_2"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "J_n"', "\shortmid"{marking}] & a_n \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[rrr, dashed, no head] & & & b
\end{tikzcd}
$$

も各点左 Kan 拡張である．φ が cocartesian だから補題 29 により

$$
\begin{tikzcd}
u \arrow[r, "J_1(k{,}\mathrm{id})", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta"] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mathrm{id}"] & \cdots \arrow[r, "J_n", "\shortmid"{marking}] & a_n \arrow[d, equal] \\
a_0 \arrow[r, "J_1"', "\shortmid"{marking}] \arrow[d, "l"'] & a_1 \arrow[r, "J_2"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "J_n"', "\shortmid"{marking}] & a_n \arrow[d, "g"] \\
b \arrow[rrr, dashed, no head] & & & b
\end{tikzcd}
$$

が各点左 Kan 拡張である． □

ここで $\mathbb{D}$ に対する次の条件を考える．

(wc) 任意の $\vec{J} = \langle a_0, J_1, \cdots, J_n, a_n \rangle$ に対して，弱 cocartesian パス

$$
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[d, "f_1"] \\
a_0 \arrow[r, "J_1"', "\shortmid"{marking}] & a_1
\end{tikzcd}
, \quad \cdots, \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
a_{n-1} \arrow[r, "J_n"', "\shortmid"{marking}] & a_n
\end{tikzcd}
$$

が存在する．

(nrc) 任意の $\vec{J} = \langle a_0, J_1, \cdots, J_n, a_n \rangle$ に対して，nullary-右 cocartesian パス

$$
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[d, "f_1"] \\
a_0 \arrow[r, "J_1"', "\shortmid"{marking}] & a_1
\end{tikzcd}
, \quad \cdots, \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
a_{n-1} \arrow[r, "J_n"', "\shortmid"{marking}] & a_n
\end{tikzcd}
$$

が存在する．

補題 41. $\mathbb{D}$ が左制限と右 cocartesian tabulation を持つならば条件 (wc) を満たす．

証明. 任意の $\vec{J}$ を取る．まず仮定より右 cocartesian tabulation

$$
\begin{tikzcd}
x_1 \arrow[r, dashed, no head] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x_1 \arrow[d, "q_1"] \\
a_0 \arrow[r, "J_1"', "\shortmid"{marking}] & a_1
\end{tikzcd}
$$

が取れる．β_1 が右 cocartesian cell なので左制限 $\langle J_2(q_1, \mathrm{id}), \gamma \rangle$ を考えると，$\langle \beta_1, \gamma \rangle$ も右 cocartesian パスである．ここで $J_2(q_1, \mathrm{id})$ の右 cocartesian tabulation $\langle x_2, \pi_2 \rangle$ を考

える．

$$
\begin{tikzcd}
x_2 \arrow[r, dashed, no head] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Downarrow Up_2"] & x_2 \arrow[r, dashed, no head] \arrow[d, "p_2"] \arrow[dr, phantom, "\Downarrow \pi_2"] & x_2 \arrow[d, "q_2"] \\
x_1 \arrow[r, dashed, no head] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Downarrow \beta_1"] & x_1 \arrow[r, "|" marking, "J_2(q_1,\mathrm{id})"'] \arrow[d, "q_1"] \arrow[dr, phantom, "\Downarrow \gamma"] & a_2 \arrow[d, equal] \\
a_0 \arrow[r, "|" marking, "J_1"'] & a_1 \arrow[r, "|" marking, "J_2"'] & a_2
\end{tikzcd}
$$

このとき命題 9 より $\langle Up_2, \pi_2 \rangle$ も右 cocartesian である．$\mathbb{D}$ が左制限を持つから補題 11（の horizontal dual）より $\langle \beta_1 \circ Up_2, \gamma \circ \pi_2 \rangle$ も右 cocartesian パスである．故に $n = 2$ の場合の条件 (wc) が証明できた．一般の n の場合はこの操作を繰り返せばよい． □

補題 42. $\mathbb{D}$ が左制限と nullary-右 cocartesian tabulation を持つならば条件 (nrc) を満たす．

証明. 補題 41 と同様にして成り立つ． □

命題 43. $\mathbb{D}$ が左制限を持ち，条件 (nrc) を満たすとする．このとき

(1) r-各点左 Kan 拡張は各点左 Kan 拡張である．
(2) r-各点左 Kan 拡張は r^+-各点左 Kan 拡張である．

証明. (1) $\mathbb{D}$ の cell

$$
\begin{tikzcd}
a_0 \arrow[r, "|" marking, "J_1"] \arrow[d, "l"'] & a_1 \arrow[r, "|" marking, "J_2"] & \cdots \arrow[r, "|" marking, "J_{n-1}"] \arrow[d, phantom, "\Downarrow \eta"] & a_{n-1} \arrow[r, "|" marking, "J_n"] & a_n \arrow[d, "g"] \\
b \arrow[rrrr, dashed, no head] & & & & b
\end{tikzcd}
$$

を r-各点左 Kan 拡張とする．これが各点左 Kan 拡張であることを示すために, 次の図式における cell θ を考える．条件 (nrc) により図式中の cell $\beta_1, \cdots, \beta_m$ を nullary-右 cocartesian パスになるように取ることができる．更に $\mathbb{D}$ が左制限を持つから $\langle J_1(f_m, \mathrm{id}), \gamma \rangle$ も考えることができて，次の図式全体が得られる．

$$
\sigma \quad := \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow \beta_1"] & x \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] \arrow[d, phantom, "\cdots"] & x \arrow[r, dashed, no head] \arrow[d, "f_{m-1}"'] \arrow[dr, phantom, "\Downarrow \beta_m"] & x \arrow[r, "|" marking, "J_1(f_m,\mathrm{id})"] \arrow[d, "f_m"] \arrow[dr, phantom, "\Downarrow \gamma"] & a_1 \arrow[r, "|" marking, "J_2"] \arrow[d, equal] & \cdots \arrow[r, "|" marking, "J_n"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & a_n \arrow[d, equal] \\
b_0 \arrow[r, "|" marking, "K_1"'] \arrow[d, "k"'] & b_1 \arrow[r, "|" marking, "K_2"'] & \cdots \arrow[r, "|" marking, "K_{m-1}"'] & b_{m-1} \arrow[r, "|" marking, "K_m"'] & a_0 \arrow[r, "|" marking, "J_1"'] \arrow[d, phantom, "\Downarrow \theta"] & a_1 \arrow[r, "|" marking, "J_2"'] & \cdots \arrow[r, "|" marking, "J_n"'] & a_n \arrow[d, "g"] \\
b \arrow[rrrrrrr, dashed, no head] & & & & & & & b
\end{tikzcd}
$$

η が r-各点左 Kan 拡張だから，次の等式を満たす τ が得られる．

$$\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] & x \arrow[r, "J_1(f_m,\mathrm{id})", "\shortmid"{marking}] \arrow[d, "f_m"'] \arrow[dr, phantom, "\Downarrow\gamma"] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "J_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mathrm{id}"] & a_n \arrow[d, equal] \\
b_0 \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "k"'] & a_0 \arrow[r, "J_1"', "\shortmid"{marking}] \arrow[d, "l"'] & a_1 \arrow[r, "J_2"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "J_n"', "\shortmid"{marking}] & a_n \arrow[d, "g"] & = \quad \sigma \\
b \arrow[r, dash, dashed] & b \arrow[rrr, dash, dashed] & & & b
\end{tikzcd}$$

この τ を次の左辺のように分解する．

$$\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dash, dashed] \arrow[d, "f_1"] & \cdots \arrow[r, dash, dashed] & x \arrow[r, dash, dashed] \arrow[d, "f_{m-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_m"] \\
b_0 \arrow[r, "K_1"', "\shortmid"{marking}] \arrow[d, "k"'] & b_1 \arrow[r, "K_2"', "\shortmid"{marking}] & \cdots \arrow[r, "K_{m-1}"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\tau'"] & b_{m-1} \arrow[r, "K_m"', "\shortmid"{marking}] & a_0 \arrow[d, "l"] \\
c \arrow[rrrr, dash, dashed] & & & & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] & x \arrow[d, "f_m"] \\
b_0 \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "k"'] & a_0 \arrow[d, "l"] \\
c \arrow[r, dash, dashed] & d
\end{tikzcd}$$

すると $\langle\beta_1,\cdots,\beta_n\rangle$ が nullary-右 cocartesian パスだから

$$\begin{tikzcd}
b_0 \arrow[r, "\vec{K}", "\shortmid\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau'"] & a_0 \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta"] & a_n \arrow[d, "g"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b_0 \arrow[r, "\vec{K}", "\shortmid\shortmid"{marking}] \arrow[d, "k"'] & a_0 \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\theta"] & a_n \arrow[d, "g"] \\
b \arrow[rr, dash, dashed] & & b
\end{tikzcd}$$

が得られる．このような τ' の一意性も分かるから η は各点左 Kan 拡張である．

(2) η を r-各点左 Kan 拡張とする．$k\colon u \to a_0$ に対して制限 $\langle J_1(k,\mathrm{id}),\beta\rangle$ が存在するとき

$$\widetilde{\eta} \quad := \quad \begin{tikzcd}
u \arrow[r, "J_1(k,\mathrm{id})", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta"] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "J_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mathrm{id}"] & a_n \arrow[d, equal] \\
a_0 \arrow[r, "J_1"', "\shortmid"{marking}] \arrow[d, "l"'] & a_1 \arrow[r, "J_2"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\eta"] & \cdots \arrow[r, "J_n"', "\shortmid"{marking}] & a_n \arrow[d, "g"] \\
b \arrow[rrr, dash, dashed] & & & b
\end{tikzcd}$$

は補題 31 により r-各点左 Kan 拡張である．よって (1) により $\widetilde{\eta}$ も各点左 Kan 拡張である． □

系 44. $\mathbb{D}$ が左制限と companion を持ち (nrc) を満たすとき，各点左 Kan 拡張と r-各点左 Kan 拡張と r$^+$-各点左 Kan 拡張は一致する．

証明. 命題 40, 43 より分かる． $\square$

命題 45. $\eta\colon g \Rightarrow l \circ f$ を $\mathcal{V}(\mathbb{D})$ の 2-morphism，即ち下の図式の右辺のような cell とする．f が conjoint を持つとして，これを使って

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] & a \arrow[dd, "g"] \\
b \arrow[d, "l"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
$$

と分解する．このとき η' が cartesian ならば，$\mathcal{V}(\mathbb{D})$ において $l_\dagger g = \langle f, \eta \rangle$ であり，更にこれは絶対左 Kan リフトである．

証明. 絶対左 Kan リフトであることを示すため，$\mathcal{V}(\mathbb{D})$ における任意の図式

$$
\begin{tikzcd}
x \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\theta\Uparrow"] & b \arrow[d, "l"] \\
a \arrow[r, "g"'] & c
\end{tikzcd}
$$

を考える．これは $\mathbb{D}$ の図式で描けば，次の右辺の cell になる．今 η' が cartesian なので，この θ は左辺のように分解できる．

$$
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta'"] & x \arrow[d, "h"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "k"'] & x \arrow[d, "h"] \\
b \arrow[d, "l"'] \arrow[r, phantom, "\Downarrow\theta"] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
$$

このとき

$$
\tau \quad := \quad
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta'"] & x \arrow[d, "h"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

と定めれば

$$
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[dd, "k"'] \arrow[ddr, phantom, "\Downarrow\tau"] & x \arrow[r, dash, dashed] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow Uh"] & x \arrow[d, "h"] \\
 & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[dd, "g"] \\
b \arrow[r, dash, dashed] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] & \\
c \arrow[r, dash, dashed] & c \arrow[r, dash, dashed] & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta'"] & x \arrow[r, dash, dashed] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow Uh"] & x \arrow[d, "h"] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c \arrow[r, dash, dashed] & c
\end{tikzcd}
= \theta
$$

である．これは $\mathcal{V}(\mathbb{D})$ の図式で描けば

$$
\begin{tikzcd}
x \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\tau\Uparrow", pos=0.2] & b \arrow[d, "l"] \\
a \arrow[ur, "f"] \arrow[r, "g"'] \arrow[ur, phantom, "\eta\Uparrow", pos=0.8] & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\theta\Uparrow"] & b \arrow[d, "l"] \\
a \arrow[r, "g"'] & c
\end{tikzcd}
$$

となる．このような τ は一意だから，絶対左 Kan リフトであることが分かった． $\square$

この命題は $\mathbb{D}$ に条件 (wc) を付けると逆も言える．

命題 46. $l_\dagger g = \langle f, \eta \rangle$ を $\mathcal{V}(\mathbb{D})$ における絶対左 Kan リフトとする．f が conjoint を持つとして，η を次のように分解する．

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[dd, "g"] \\
b \arrow[d, "l"'] & \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
$$

更に条件 (wc) を満たすとする．このとき η' は cartesian である．

証明. cartesian であることを示すため，次のような cell θ を考える．

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid\shortmid"{marking}] \arrow[d, "h"'] \arrow[ddr, phantom, "\Downarrow\theta"] & v \arrow[d, "k"] \\
b \arrow[d, "l"'] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
$$

(wc) により弱 cocartesian パス $\langle \beta_1, \cdots, \beta_n \rangle$ を取って

$$
\sigma \quad := \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & x \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}"', "\shortmid" marking] & a_{n-1} \arrow[r, "J_n"', "\shortmid" marking] & v \arrow[d, "k"] \\
b \arrow[d, "l"'] & & \Downarrow\theta & & a \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

とすれば，$l_\dagger g = \langle f, \eta \rangle$ が絶対左 Kan リフトだから次の τ が得られる．

$$
\begin{tikzcd}
x \arrow[r, "h \circ f_0"] \arrow[d, "k \circ f_n"'] & b \arrow[d, "l"] \\
a \arrow[ur, "f"] \arrow[r, "g"'] & c
\end{tikzcd}
\;(\tau \Uparrow,\ \eta \Uparrow)
\quad = \quad
\begin{tikzcd}
x \arrow[r, "h \circ f_0"] \arrow[d, "k \circ f_n"'] \arrow[dr, phantom, "\sigma \Uparrow"] & b \arrow[d, "l"] \\
a \arrow[r, "g"'] & c
\end{tikzcd}
$$

これは $\mathbb{D}$ の図式で描けば

$$
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & x \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}"', "\shortmid" marking] & a_{n-1} \arrow[r, "J_n"', "\shortmid" marking] & v \arrow[d, "k"] \\
b \arrow[d, "l"'] & & \Downarrow\theta & & a \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\tau"] & x \arrow[r, dashed, no head] \arrow[d, "k \circ f_n"'] \arrow[dr, phantom, "\Downarrow U(k \circ f_n)"] & x \arrow[d, "k \circ f_n"] \\
u \arrow[d, "h"'] & a \arrow[r, dashed, no head] \arrow[d, "f"] & a \arrow[dd, "g"] \\
b \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \arrow[ur, phantom, "\Downarrow\eta"] & \\
c \arrow[r, dashed, no head] & c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\tau"] & x \arrow[r, dashed, no head] \arrow[d, "k \circ f_n"'] \arrow[dr, phantom, "\Downarrow U(k \circ f_n)"] & x \arrow[d, "k \circ f_n"] \\
u \arrow[d, "h"'] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "l"'] & b \arrow[r, "f^\star"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[rr, dashed, no head] & & c
\end{tikzcd}
$$

となる．このとき $\langle \beta_1, \cdots, \beta_n \rangle$ が弱 cocartesian だから次の等式を満たす θ' が存在する．

$$
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dash, dashed] \arrow[d, "f_1"] & \cdots \arrow[r, dash, dashed] & x \arrow[r, dash, dashed] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\theta'"] & a_{n-1} \arrow[r, "J_n"', "\shortmid" marking] & v \arrow[d, "k"] \\
b \arrow[rrrr, "f^\star"', "\shortmid" marking] & & & & a
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] & x \arrow[r, dash, dashed] \arrow[d, "k\circ f_n"'] \arrow[dr, phantom, "\Downarrow U(k\circ f_n)"] & x \arrow[d, "k\circ f_n"] \\
u \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "h"'] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] & b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
$$

すると

$$
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dash, dashed] \arrow[d, "f_1"] & \cdots \arrow[r, dash, dashed] & x \arrow[r, dash, dashed] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\theta'"] & a_{n-1} \arrow[r, "J_n"', "\shortmid" marking] & v \arrow[d, "k"] \\
b \arrow[rrrr, "f^\star"', "\shortmid" marking] \arrow[d, "l"'] & & {} \arrow[d, phantom, "\Downarrow\eta'"] & & a \arrow[d, "g"] \\
c \arrow[rrrr, dash, dashed] & & {} & & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] & x \arrow[r, dash, dashed] \arrow[d, "k\circ f_n"'] \arrow[dr, phantom, "\Downarrow U(k\circ f_n)"] & x \arrow[d, "k\circ f_n"] \\
u \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "h"'] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "l"'] & b \arrow[r, "f^\star"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[rr, dash, dashed] & {} & c
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
x \arrow[r, dash, dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dash, dashed] \arrow[d, "f_1"] & \cdots \arrow[r, dash, dashed] & x \arrow[r, dash, dashed] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "\shortmid" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "\shortmid" marking] & \cdots \arrow[r, "J_{n-1}"', "\shortmid" marking] & a_{n-1} \arrow[r, "J_n"', "\shortmid" marking] & v \arrow[d, "k"] \\
b \arrow[d, "l"'] & & \Downarrow\theta & & a \arrow[d, "g"] \\
c \arrow[rrrr, dash, dashed] & & & & c
\end{tikzcd}
$$

となるから，$\langle \beta_1, \cdots, \beta_n \rangle$ の普遍性により

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & v \arrow[d, "k"] \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, "h"'] & v \arrow[d, "k"] \\
b \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "l"'] & a \arrow[d, "g"] \\
c \arrow[r, dash, dashed] & c
\end{tikzcd}
$$

である．θ' の一意性も容易に分かるから η' は cartesian である． $\square$

5　米田射

定義. 垂直射 $f\colon a \to b$ が**忠実充満** $\iff$ cell

$$\begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が cartesian である.

例 47. (V, V')-$\mathbb{P}$ROF における垂直射 $F\colon \mathcal{C} \to \mathcal{D}$ が忠実充満となるのは F が V'-忠実充満のときである. □

　これは次のようにして 2-category における忠実充満と関係がある.

命題 48. 垂直射 $f\colon a \to b$ に対して

$$f \text{ が } \mathbb{D} \text{ において忠実充満} \Longrightarrow f \text{ が } \mathcal{V}(\mathbb{D}) \text{ において忠実充満}^{*9}$$

である. 条件 (wc) が成り立つならば逆も成り立つ.

証明. $(\Longrightarrow)$ $f\colon a \to b$ が忠実充満だとすると, Uf が cartesian なので

$$\begin{tikzcd} c \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & c \arrow[d, "g"] \\ a \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd} = \begin{tikzcd} c \arrow[r, dashed, no head] \arrow[d, "h"'] & c \arrow[d, "g"] \\ a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

により $\mathcal{V}(\mathbb{D})(c,a)(g,h) \ni \theta' \xrightarrow{f\bullet -} \theta \in \mathcal{V}(\mathbb{D})(c,a)(f \circ g, f \circ h)$ が全単射になる. 即ち f が $\mathcal{V}(\mathbb{D})$ において忠実充満である.

*9 「2-category での随伴・Kan 拡張・忠実充満」の PDF を参照.

($\Longleftarrow$) Uf が cartesian であることを示すため，次のような cell θ を考える．

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}", "||" marking] \arrow[d, "h"'] & v \arrow[d, "g"] \\
a \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow\theta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

条件 (wc) により弱 cocartesian パス $\langle \beta_1, \cdots, \beta_n \rangle$ を取って

$$
\sigma \quad \coloneqq \quad
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & x \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "|" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "|" marking] & \cdots \arrow[r, "J_{n-1}"', "|" marking] & a_{n-1} \arrow[r, "J_n"', "|" marking] & v \arrow[d, "g"] \\
a \arrow[d, "f"'] & & \Downarrow\theta & & a \arrow[d, "f"] \\
b \arrow[rrrr, dashed, no head] & & & & b
\end{tikzcd}
$$

とすれば，f が $\mathcal{V}(\mathbb{D})$ において忠実充満だから，$\sigma' \colon g \circ f_n \Rightarrow h \circ f_0$ を使って $\sigma = f \bullet \sigma'$ と書ける．更に $\langle \beta_1, \cdots, \beta_n \rangle$ が弱 cocartesian パスだから

$$
\begin{tikzcd}
x \arrow[r, dashed, no head] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & x \arrow[r, dashed, no head] \arrow[d, "f_1"] & \cdots \arrow[r, dashed, no head] & x \arrow[r, dashed, no head] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & x \arrow[d, "f_n"] \\
u \arrow[r, "J_1"', "|" marking] \arrow[d, "h"'] & a_1 \arrow[r, "J_2"', "|" marking] & \cdots \arrow[r, "J_{n-1}"', "|" marking] \arrow[d, phantom, "\Downarrow\sigma''"] & a_{n-1} \arrow[r, "J_n"', "|" marking] & v \arrow[d, "g"] \\
a \arrow[rrrr, dashed, no head] & & & & a
\end{tikzcd}
\quad = \quad \sigma'
$$

とできる．このとき明らかに

$$
\begin{tikzcd}
u \arrow[r, "\vec{J}", "||" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\sigma''"] & v \arrow[d, "g"] \\
a \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[r, "\vec{J}", "||" marking] \arrow[d, "h"'] & v \arrow[d, "g"] \\
a \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow\theta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

である．このような σ'' が一意であることも分かる． $\square$

次に稠密性を定義する．

定義. 垂直射 $f\colon a \to b$ が**稠密** $\iff$ cell

$$\begin{tikzcd} u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が cartesian ならば左 Kan 拡張である．

稠密の定義には左 Kan 拡張ではなく r-各点左 Kan 拡張を使っても同値になる．即ち次の命題が成り立つ．

命題 49. 垂直射 $f\colon a \to b$ が稠密 $\iff$ cell

$$\begin{tikzcd} u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が cartesian ならば r-各点左 Kan 拡張である．

証明. ($\Longleftarrow$) 定義から明らか.

($\Longrightarrow$) 次の図式で f が稠密，η が cartesian であるとする．

$$\begin{tikzcd} u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

η が r-各点左 Kan 拡張であることを示すため，$k\colon v \to u$ を取り制限 $\langle J(k, \mathrm{id}), \beta\rangle$ が存在するとする．このとき補題 5 より

$$\begin{tikzcd} v \arrow[r, "J(k{,}\mathrm{id})", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\ u \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

も cartesian である．よって f が稠密だからこれは左 Kan 拡張である． $\square$

f が conjoint を持つ場合は，稠密性は ε_f で判定できる．

命題 50. $f\colon a \to b$ の conjoint $f^\star$ が存在するとき

$$f \text{ が稠密である} \iff \varepsilon_f \text{ が r-各点左 Kan 拡張である．}$$

$$\begin{tikzcd} b \arrow[r, "f^\star", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

証明. ($\Longrightarrow$) ε_f が cartesian だから，命題 49 により ε_f は r-各点左 Kan 拡張になる．

($\Longleftarrow$) f が稠密であることを示すため，次の右辺のような cartesian cell η を取る．これを左辺のように分解して η' を得る．

$$\begin{tikzcd} u \arrow[r, "J", "|"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, equal] \\ b \arrow[r, "f^\star", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd} = \begin{tikzcd} u \arrow[r, "J", "|"{marking}] \arrow[dd, "l"'] \arrow[ddr, phantom, "\Downarrow\eta"] & a \arrow[dd, "f"] \\ & \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

補題 5 より η' も cartesian である．よって仮定より ε_f が r-各点左 Kan 拡張だから，η も左 Kan 拡張になる． □

従って $\mathbb{D}$ が左制限と nullary-弱 cocartesian tabulation を持つとして $f\colon a \to b$ の conjoint が存在するならば，命題 35 により

$$f \text{ が稠密} \iff \mathcal{V}(\mathbb{D}) \text{ において } f^\dagger f = \langle \mathrm{id}, \mathrm{id} \rangle \text{ が c-各点左 Kan 拡張}$$

となる．

稠密の定義で左 Kan 拡張を各点左 Kan 拡張にすることで次の定義を得る．

定義. 垂直射 $f\colon a \to b$ が**各点稠密** $\iff$ cell

$$\begin{tikzcd} u \arrow[r, "J", "|"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が cartesian ならば各点左 Kan 拡張である．

各点稠密に対しても稠密の場合と同様にして以下の命題が証明できる.

命題 51. 垂直射 $f\colon a \to b$ が各点稠密 $\iff$ cell

$$\begin{tikzcd} u \arrow[r, "J", "|"] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が cartesian ならば r$^+$-各点左 Kan 拡張である. □

命題 52. $f\colon a \to b$ の conjoint $f^\star$ が存在するとき

$$f \text{ が各点稠密である} \iff \varepsilon_f \text{が r}^+\text{-各点左 Kan 拡張である.}$$ □

系 53. $\mathbb{D}$ が左制限を持ち,条件 (nrc) を満たすとき,稠密と各点稠密は同値である.

証明. 命題 43 から分かる. □

定義. 垂直射 $f\colon a \to \widehat{a}$ が**米田射**
$\iff$ 任意の水平射 $J\colon u \nrightarrow a$ に対して,垂直射 $l\colon u \to \widehat{a}$ と cartesian cell

$$\begin{tikzcd} u \arrow[r, "J", "|"] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\sigma_f^J"] & a \arrow[d, "f"] \\ \widehat{a} \arrow[r, dashed, no head] & \widehat{a} \end{tikzcd}$$

が存在する.

米田射 f が稠密ならば,$J\colon a \nrightarrow b$ に対する cartesian cell σ_f^J は左 Kan 拡張である(命題 49 によればより強く r-各点左 Kan 拡張である).よってこの場合の l は同型を除いて一意に定まり,$l \cong J^\dagger f$ となる.

定義. 忠実充満かつ稠密な米田射を**米田埋込**といい,忠実充満かつ各点稠密な米田射を**各点米田埋込**という.

米田埋込 $y\colon a \to \widehat{a}$ に対して $y^\star$ が存在するならば,命題 50 より $y^{\star\dagger}y \cong \mathrm{id}$ である.

例 54. 例 2 の (V, V')-$\mathbb{P}$ROF を考える.$\mathcal{C}$ を V-豊穣圏とすると $\widehat{\mathcal{C}} \coloneqq [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ は V'-豊穣圏として定義できて,(豊穣圏論における)米田埋込 $y\colon \mathcal{C} \to \widehat{\mathcal{C}}$ は (V, V')-$\mathbb{P}$ROF の垂直射になるのだった.この y は (V, V')-$\mathbb{P}$ROF における各点米田埋込である.

証明.（忠実充満）y は V'-忠実充満だから，例 47 より (V,V')-$\mathbb{P}$ROF における忠実充満である．

（各点稠密）cartesian cell

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Phi", "|" marking] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[d, "y"] \\
\widehat{\mathcal{C}} \arrow[r, dashed, no head] & \widehat{\mathcal{C}}
\end{tikzcd}$$

を取ると $\eta_{cb}\colon \Phi(c,b) \to \widehat{\mathcal{C}}(y(c), Lb) \cong L(c,b)$ は同型である．η が各点左 Kan 拡張であることを示すため，次の右辺の θ を取る．

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\vec{\Psi}", "||" marking] \arrow[d, "K"'] \arrow[dr, phantom, "\Downarrow\tau"] & \mathcal{B} \arrow[r, "\Phi", "|" marking] \arrow[d, "L"] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[d, "y"] \\
\widehat{\mathcal{C}} \arrow[r, dashed, no head] & \widehat{\mathcal{C}} \arrow[r, dashed, no head] & \widehat{\mathcal{C}}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{A} \arrow[r, "\vec{\Psi}", "||" marking] \arrow[d, "K"'] \arrow[drr, phantom, "\Downarrow\theta"] & \mathcal{B} \arrow[r, "\Phi", "|" marking] & \mathcal{C} \arrow[d, "y"] \\
\widehat{\mathcal{C}} \arrow[rr, dashed, no head] & & \widehat{\mathcal{C}}
\end{tikzcd}$$

ここでは簡単のため $|\vec{\Psi}| = 1$ として $\vec{\Psi} = \langle \mathcal{A}, \Psi, \mathcal{B} \rangle$ と書く．即ち $\theta_{abc}\colon \Psi(b,a) \otimes \Phi(c,b) \to \widehat{\mathcal{C}}(y(c), Ka) \cong K(c,a)$ である．そこで

$$\tau_{abc} \coloneqq \left(\Psi(b,a) \xrightarrow{\widetilde{\theta}_{abc}} [\Phi(c,b), K(c,a)] \xrightarrow{[\eta_{cb}^{-1}, \mathrm{id}]} [L(c,b), K(c,a)] \right)$$

とするとこれは c について自然だから $\tau_{ab}\colon \Psi(b,a) \to \widehat{\mathcal{C}}(Lb, Ka)$ を定める．即ち上記図式の左辺の cell τ になる．すると定義から上記の等式が成り立つ．このような τ は一意だから η は各点左 Kan 拡張である．

（米田射）$\Phi\colon \mathcal{B} \nrightarrow \mathcal{C}$ を V-profunctor とする．V'-関手 $\Phi^\dagger y\colon \mathcal{B} \to \widehat{\mathcal{C}}$ を $\Phi^\dagger y(b) \coloneqq \Phi(b)$ により定める．このとき cell

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Phi", "|" marking] \arrow[d, "\Phi^\dagger y"'] \arrow[dr, phantom, "\Downarrow\beta"] & \mathcal{C} \arrow[d, "y"] \\
\widehat{\mathcal{C}} \arrow[r, dashed, no head] & \widehat{\mathcal{C}}
\end{tikzcd}$$

とは $\beta_{cb}\colon \Phi(c,b) \to \widehat{\mathcal{C}}(y(c), \Phi(b))$ のことである．$\widehat{\mathcal{C}}(y(c), \Phi(b)) \cong \Phi(c,b)$ だから，これにより定まる cell を σ_y^J とすれば σ_y^J は cartesian である． □

※ この例で V'-$\mathbb{P}$ROF ではなく (V,V')-$\mathbb{P}$ROF を考えているのは，米田射であることを示す際の $\Phi^\dagger\colon \mathcal{B} \to \widehat{\mathcal{C}}$ が，$\Phi\colon \mathcal{B} \nrightarrow \mathcal{C}$ が V-profunctor でなければ定義できないからである．つまり V'-$\mathbb{P}$ROF は水平射として V'-profunctor を含んでおり，これは「多すぎる」．

定理 55. $y\colon a \to \widehat{a}$ を米田埋込として，$f\colon a \to b$ の conjoint $f^\star$ が存在するとき

$$\text{次の図式が } \mathcal{V}(\mathbb{D})(a, \widehat{a}) \text{ の同型} \Longrightarrow f \text{ は忠実充満}$$

$$\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "|" marking] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

である．制限 $b(f,f)$ が存在するならば逆も成り立つ．

証明. $(\Longrightarrow)$ y が忠実充満だから Uy は cartesian である．よって Uy と同型な cell を合成した

$$\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow Uy"] & a \arrow[dd, "y"] \\
b \arrow[r, "f^\star", "|" marking] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] & \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

も cartesian である．$\sigma_y^{f^\star}$ が cartesian だから補題 5 により η_f も cartesian になる．よって再び補題 5 により，次の左辺の合成も cartesian である．

$$\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}$$

従って f は忠実充満である．

$(\Longleftarrow)$ 制限 $\langle b(f,f), \beta \rangle$ が存在するとして

$$\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f{,}f)", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}$$

と分解する．f が忠実充満だから Uf は cartesian である．故に補題 5 から γ も cartesian である．更にこの γ は nullary-弱 cocartesian である．

∵) まず γ が cartesian だから

$$
\begin{tikzcd}
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)"', "\shortmid"{marking}] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{b(f,f)}"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)"', "\shortmid"{marking}] & a
\end{tikzcd}
$$

と分解できる．このとき

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)"', "\shortmid"{marking}] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{b(f,f)}"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)"', "\shortmid"{marking}] & a
\end{tikzcd}
= \gamma =
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)"', "\shortmid"{marking}] & a
\end{tikzcd}
$$

となるから γ の普遍性により

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] & a
\end{tikzcd}
$$

となる．

　これを踏まえて γ が nullary-弱 cocartesian であることを示すため次の右辺の cell

θ を考える．θ' を

$$
\theta' \quad \coloneqq \quad
\begin{tikzcd}
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta"] & a \arrow[d, "k"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
$$

により定義すれば

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & a \arrow[d, "k"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
a \arrow[r, dashed, no head] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta"] & a \arrow[d, "k"] \\
c \arrow[r, dashed, no head] & c
\end{tikzcd}
\quad = \quad \theta
$$

となる．このような θ' の一意性も分かるから γ は nullary-弱 cocartesian である．

次に β を

$$
\begin{tikzcd}
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta'"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

と分解すれば

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta'"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

となるから，ε_f の普遍性により

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
a \arrow[r, "b(f,f)", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta'"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] & a
\end{tikzcd}
$$

となる．よって今考えたい次の左辺の図式は，右辺のようになる．

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] & \\
a \arrow[r, "b(f,f)", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta'"] & a \arrow[d, equal] & \\
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] & \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} &
\arrow[from=2-2, to=4-2, phantom, shift left=6ex, "\}(a)"]
\arrow[from=1-2, to=4-2, phantom, shift left=12ex, "\}(b)"]
\end{tikzcd}
$$

y が稠密だから $\sigma_y^{f^\star}$ は r-各点左 Kan 拡張である．β' は cartesian だったから (a) の部分は左 Kan 拡張となる．γ は nullary-弱 cocartesian だったから補題 22 により (b) も左 Kan 拡張である．従って命題 20 からこれは $\mathcal{V}(\mathbb{D})(a,\widehat{a})$ の同型射となる． $\square$

$f\colon a\to b$ を垂直射，$u\in\mathbb{D}$ を対象とする．任意の $g\colon u\to b$ に対して制限 $\langle b(g,f),\beta_g\rangle$ が存在するとする．このとき $g\mapsto b(g,f)$ は関手 $b(-,f)\colon\mathcal{V}(\mathbb{D})(u,b)\to\mathcal{H}(\mathbb{D})(a,u)$ を定める．

∵) $\mathcal{V}(\mathbb{D})(u,b)$ の射

$$
\begin{tikzcd}
u \arrow[r, dash, dashed] \arrow[d, "g'"'] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[d, "g"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

に対して $\mathcal{H}(\mathbb{D})(a,u)$ の射 $b(\tau,f)\colon b(g,f)\to b(g',f)$ を次の等式で定める．

$$
\begin{tikzcd}
u \arrow[rr, "b(g{,}f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\Downarrow b(\tau{,}f)"] & & a \arrow[d, equal] \\
u \arrow[rr, "b(g'{,}f)", "\shortmid"{marking}] \arrow[d, "g'"'] \arrow[rrd, phantom, "\Downarrow\beta_{g'}"] & & a \arrow[d, "f"] \\
b \arrow[rr, dashed, no head] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[r, dashed, no head] \arrow[dd, "g'"'] \arrow[rdd, phantom, "\Downarrow\tau"] & u \arrow[r, "b(g{,}f)", "\shortmid"{marking}] \arrow[dd, "g"] \arrow[rdd, phantom, "\Downarrow\beta_g"] & a \arrow[dd, "f"] \\
 & & \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

普遍性によりこれは関手を定める．

この関手 $b(-,f)$ により f の性質を次のように言い換えることができる．

命題 56. 以下の同値が成り立つ．

(1) f が稠密 $\Longleftrightarrow$ 任意の $u\in\mathbb{D}$ に対して $b(-,f)$ が忠実充満である．

(2) f が米田射 $\Longleftrightarrow$ 任意の $u\in\mathbb{D}$ に対して $b(-,f)$ が本質的全射である．

証明. ((1) の $\Longrightarrow$) f が稠密とすると β_g が左 Kan 拡張となるから，普遍性により γ と $b(\gamma,f)$ は 1 対 1 に対応する．

((1) の $\Longleftarrow$) $b(-,f)$ が忠実充満であるとして

$$
\begin{tikzcd}
u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[rd, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

を cartesian とする．普遍性により $\mathcal{H}(\mathbb{D})(u,a)$ の同型 $b(l,f)\cong J$ が成り立つので，この同型を σ とする．

η が左 Kan 拡張であることを示すため，次の右辺のような θ を取る．このとき β_k の普遍性により左辺の θ' が得られる．

$$
\begin{tikzcd}
u \arrow[r, "b(l{,}f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\Downarrow\theta'"] & a \arrow[d, equal] \\
u \arrow[r, "b(k{,}f)", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[rd, phantom, "\Downarrow\beta_k"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[r, "b(l{,}f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[rd, phantom, "\Downarrow\theta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

$b(-,f)$ が忠実充満だから $b(\tau,f)=\theta'$ とできる．即ち

$$
\begin{tikzcd}
u \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[r, phantom, shift right=8ex, "\Downarrow\tau"] & u \arrow[r, "b(l,f)", "\shortmid"{marking}] \arrow[d, "l"] \arrow[r, phantom, shift right=8ex, "\Downarrow\beta_g"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "b(l,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\theta'"] & a \arrow[d, equal] \\
u \arrow[r, "b(k,f)", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\beta_k"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "b(l,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, equal] \\
u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

が成り立つ．よって

$$
\begin{tikzcd}
u \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[r, phantom, shift right=8ex, "\Downarrow\tau"] & u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "l"] \arrow[r, phantom, shift right=8ex, "\Downarrow\eta"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma^{-1}"] & a \arrow[d, equal] \\
u \arrow[r, dash, dashed] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[r, "b(l,f)", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\beta_l"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
= \quad \theta
$$

となる．このような τ は一意だから η は左 Kan 拡張である．

(2) 米田射の定義から明らか． □

6　米田構造との関係

$\mathbb{D}$ を augmented virtual double category として，$\mathbb{D}$ の垂直射からなる集まり A で，次の条件を満たすものを考える．

$$f\colon a\to b,\ \ g\colon b\to c,\ \ g\in A \text{ ならば } g\circ f\in A \text{ である．}$$

これはつまり米田構造のときに考えた A である（「米田構造」の PDF を参照）．そこで以下では $f\in A$ であることを f は **admissible** であるという．米田構造のときと同様，admissible であることを強調するために赤字で表すことがある．

定理 57. $\mathbb{D}$ は左制限と右 cocartesian tabulation を持つとする．admissible な $f\colon a\to b$ は conjoint $f^{\star}$ を持つとする．$y_a\colon a\to\widehat{a}$ が $\mathcal{V}(\mathbb{D})$ の条件 (y5) を満たす米田構造を定めるとして，また $J\colon u\nrightarrow a$ が水平射で a が admissible のとき，admissible な $f\colon a\to b$ と

cartesian cell

$$\begin{tikzcd} u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

が存在するとする．このとき各 y_a は $\mathbb{D}$ の各点米田埋込である．

証明. まず y_a が忠実充満であることを示す．$\mathbb{D}$ が左制限と右 cocartesian tabulation を持つので補題 41 より条件 (wc) が成り立つ．米田構造の一般論より y_a は $\mathcal{V}(\mathbb{D})$ における忠実充満な 1-morphism だから，命題 48 により y_a は垂直射として忠実充満である．

次に y_a が各点稠密であることを示す．条件 (y5) より $y_a^\dagger y_a = \langle \mathrm{id}, \mathrm{id} \rangle$ は c-各点左 Kan 拡張である．故に命題 35 より ε_{y_a} は r-各点左 Kan 拡張である．

$$\begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "y_a"'] \arrow[dr, phantom, "\Downarrow\eta_{y_a}"] & a \arrow[d, equal] \\ \widehat{a} \arrow[r, "y_a^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_a}"] & a \arrow[d, "y_a"] \\ \widehat{a} \arrow[r, dashed, no head] & \widehat{a} \end{tikzcd} = \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "y_a"'] & a \arrow[dd, "y_a"] \\ \widehat{a} \arrow[d, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow Uf"] & \\ \widehat{a} \arrow[r, dashed, no head] & \widehat{a} \end{tikzcd}$$

$\mathbb{D}$ が左制限と右 cocartesian tabulation を持つので補題 42 より条件 (nrc) が成り立つ．よって命題 43 より ε_{y_a} は r^+-各点左 Kan 拡張である．従って命題 50 より y_a は各点稠密である．

最後に y_a が米田射であることを示す．そのために $J\colon u \nrightarrow a$ を水平射とする．仮定より admissible な $f\colon a \to b$ と右辺の cartesian cell η が存在する．これを左辺のように分解すれば，補題 5 より η' も cartesian である．

$$\begin{tikzcd} u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, equal] \\ b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd} = \begin{tikzcd} u \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ b \arrow[r, dashed, no head] & b \end{tikzcd}$$

次に米田構造によって得られる $f^\dagger y_a = \langle b(f,1), \chi^f \rangle$ を

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] \arrow[d, "{b(f,1)}"'] \arrow[dr, phantom, "\Downarrow\chi'"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, "y_a"] \\
b \arrow[r, phantom, "\Downarrow\chi^f"] \arrow[d, "{b(f,1)}"'] & {} \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
$$

のように分解する．最初に注意した通り条件 (wc) が成り立ち，また米田構造の定義より $b(f,1)_\dagger y_a = \langle f, \chi^f \rangle$ は絶対左 Kan リフトだから，命題 46 により χ' は cartesian である．よって補題 5 より合成

$$
\begin{tikzcd}
u \arrow[r, "J", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid" marking] \arrow[d, "{b(f,1)}"'] \arrow[dr, phantom, "\Downarrow\chi'"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
$$

は cartesian である． □

補題 58. $\mathbb{D}$ が左制限を持つとき，補題 16 における γ は nullary-r-左 cocartesian である．

証明. $k\colon x \to a$ を任意に取る．$\mathbb{D}$ が左制限を持つので制限 $\langle f_\star(k,\mathrm{id}), \theta \rangle$ と $\langle K(k,\mathrm{id}), \sigma \rangle$ が存在する．このとき γ' を

$$
\begin{tikzcd}
x \arrow[r, "\vec{J}'", "\shortmid\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma'"] & c \arrow[d, equal] \\
x \arrow[r, "{K(k,\mathrm{id})}", "\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & c \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid" marking] & c
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "{f_\star(k,\mathrm{id})}", "\shortmid" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[r, "\vec{J}", "\shortmid\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & d \arrow[r, "g^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"', "\shortmid" marking] \arrow[d, equal] & b \arrow[r, "\vec{J}"', "\shortmid\shortmid" marking] & d \arrow[r, "g^\star"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\gamma"] & c \arrow[d, equal] \\
a \arrow[rrr, "K"', "\shortmid" marking] & & {} & c
\end{tikzcd}
$$

により定めたとき，γ' が cocartesian であることを示せばよい．まず

$$
\tau \quad := \quad
\begin{tikzcd}
x \arrow[r, "K(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & c \arrow[d, equal] \\
a \arrow[r, "K"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[r, "\vec{J}"'] & d
\end{tikzcd}
$$

と定義すると，σ と β が cartesian だから補題 5 により τ も cartesian である．次に補題 16 の条件を使うと

$$
\begin{tikzcd}
x \arrow[r, "\vec{J}'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma'"] & c \arrow[d, equal] \\
x \arrow[r, "K(k,\mathrm{id})"] \arrow[d, "k"'] & c \arrow[dd, equal] \\
a \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "f"'] & c \arrow[d, "g"] \\
b \arrow[r, "\vec{J}"'] & d
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "\vec{J}'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma'"] & c \arrow[d, equal] \\
x \arrow[r, "K(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & c \arrow[d, equal] \\
a \arrow[r, "K"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[r, "\vec{J}"'] & d
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, "f_\star(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[r, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & d \arrow[r, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, equal] & b \arrow[r, "\vec{J}"'] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[r, "g^\star"'] & c \arrow[d, equal] \\
a \arrow[rrr, "K"'] \arrow[d, "f"'] & & \arrow[d, phantom, "\Downarrow\beta"] & c \arrow[d, "g"] \\
b \arrow[rrr, "\vec{J}"'] & & & d
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
x \arrow[r, "f_\star(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[r, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & d \arrow[r, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & c \arrow[d, equal] \\
a \arrow[r, "f_\star"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[r, "\vec{J}"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "g^\star"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, "\vec{J}"'] & d \arrow[r, dashed, no head] & d
\end{tikzcd}
$$

となる．θ と ${}_f\varepsilon$ は cartesian だから，補題 5 により

$$\begin{tikzcd}
x \arrow[r, "|" marking, "f_\star(k,\mathrm{id})"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, equal] \\
a \arrow[r, "|" marking, "f_\star"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}$$

も cartesian である．故にこれは $f \circ k$ の companion を定める．つまり γ' は

$$\begin{tikzcd}
x \arrow[r, "|" marking, "(f\circ k)_\star"] \arrow[d, equal] & b \arrow[r, "\shortparallel" marking, "\vec{J}"] \arrow[d, phantom, "\Downarrow\gamma'"] & d \arrow[r, "\shortparallel" marking, "g^\star"] & c \arrow[d, equal] \\
x \arrow[rrr, "|" marking, "K(k,\mathrm{id})"'] \arrow[d, "f\circ k"'] & & & c \arrow[d, "g"] \\
b \arrow[rrr, "\shortparallel" marking, "\vec{J}"'] \arrow[rrru, phantom, "\Downarrow\tau"] & & & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "|" marking, "(f\circ k)_\star"] \arrow[d, "f\circ k"'] \arrow[dr, phantom, "\Downarrow{}_{f\circ k}\varepsilon"] & b \arrow[r, "\shortparallel" marking, "\vec{J}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{\vec{J}}"] & d \arrow[r, "\shortparallel" marking, "g^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & c \arrow[d, "g"] \\
b \arrow[r, dashed, no head] & b \arrow[r, "\shortparallel" marking, "\vec{J}"'] & d \arrow[r, dashed, no head] & d
\end{tikzcd}$$

を満たすから補題 12 の証明により γ' は cocartesian である． □

定理 59. $\mathbb{D}$ は左制限を持つとして，admissible な $f\colon a \to b$ は conjoint $f^\star\colon a \nrightarrow b$ を持つとする．また admissible な対象 a に対して admissible な米田埋込 $y_a\colon a \to \widehat{a}$ が与えられているとする．このとき y_a は $\mathcal{V}(\mathbb{D})$ の米田構造を定める．

更に $\mathbb{D}$ が nullary-右 cocartesian tabulation を持つならば条件 (y5) を満たす．

証明. $a \in \mathbb{D}$ と $f\colon a \to b$ を admissible とする．仮定より conjoint $f^\star\colon a \nrightarrow b$ が存在するので，これを使って $b(f,1) \coloneqq f^{\star\dagger}y_a$ として

$$\chi^f \quad \coloneqq \quad \begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "|" marking, "f^\star"] \arrow[d, "b(f,1)"'] \arrow[dr, phantom, "\Downarrow\sigma_{y_a}^{f^\star}"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

と定める．これが $\mathcal{V}(\mathbb{D})$ の米田構造を定めることを示す．

(y1) $f^\dagger y_a = \langle b(f,1), \chi^f\rangle$ である．

∵) y_a が米田埋込だから $\sigma_{y_a}^{f^\star}$ は左 Kan 拡張である．よって定理 19 から分かる．

(y2) $b(f,1)_\dagger y_a = \langle f, \chi^f \rangle$ であり，これは絶対左 Kan リフトである．

$\because$) $\sigma^{f^\star}_{y_a}$ が cartesian だから命題 45 より明らか．

(y3) admissible な $a \in \mathbb{D}$ に対して $y_a^\dagger y_a = \langle \mathrm{id}, \mathrm{id} \rangle$ である．

$\because$) $y_a \colon a \to \widehat{a}$ が admissible なので，conjoint $(y_a)^\star \colon a \to \widehat{a}$ が存在する．即ち

$$
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "y_a"'] \arrow[dr, phantom, "\Downarrow\eta_{y_a}"] & a \arrow[d, equal] \\
\widehat{a} \arrow[r, "(y_a)^\star"', "|"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_a}"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "y_a"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
$$

となる．命題 3 より ε_{y_a} は cartesian だから，y_a の稠密性から ε_{y_a} は左 Kan 拡張になる．このとき定理 19 により $y_a^\dagger y_a = \langle \mathrm{id}, \mathrm{id} \rangle$ である．

(y4) $f \colon a \to b$，$g \colon b \to c$ を 1-morphism として，a, b, g が admissible のとき，$c(g \circ f, 1) \cong f^{-1} \circ c(g,1)$ である．即ち次の θ が $\mathcal{V}(\mathbb{D})$ における左 Kan 拡張になる．

$$
\theta \quad := \quad
\begin{tikzcd}
c \arrow[dr, "{c(g,1)}"] & \\
b \arrow[u, "g"] \arrow[r, "y_b"'] \arrow[ur, phantom, "\chi^g \Uparrow"] & \widehat{b} \arrow[d, "{f^{-1} := \widehat{b}(y_b \circ f, 1)}"] \\
a \arrow[u, "f"] \arrow[r, "y_a"'] \arrow[ur, phantom, "\chi^{y_b \circ f} \Uparrow"] & \widehat{a}
\end{tikzcd}
$$

∵) θ を $\mathbb{D}$ の図式として描くと

$$
\theta \quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dash, dashed] \arrow[d, "f"] & a \arrow[dddd, "y_a"] \\
b \arrow[r, dash, dashed] \arrow[d, "g"'] & b \arrow[dd, "y_b"] \arrow[ddr, phantom, "\Downarrow\chi^{y_b\circ f}"] & \\
c \arrow[d, "c(g{,}1)"'] \arrow[r, phantom, "\Downarrow\chi^{g}"] & & \\
\widehat{b} \arrow[r, dash, dashed] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "\Downarrow U(f^{-1})"] & \widehat{b} \arrow[d, "f^{-1}"] & \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dash, dashed] \arrow[d, "f"] & a \arrow[ddd, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & b \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta_{y_b\circ f}"] & \\
c \arrow[r, "g^\star"', "\shortmid" marking] \arrow[d, "c(g{,}1)"'] \arrow[dr, phantom, "\Downarrow\sigma^{g^\star}_{y_b}"] & b \arrow[d, "y_b"] & \\
\widehat{b} \arrow[r, dash, dashed] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "\Downarrow U(f^{-1})"] & \widehat{b} \arrow[r, "(y_b\circ f)^\star", "\shortmid" marking] \arrow[d, "f^{-1}"] \arrow[dr, phantom, "\Downarrow\sigma^{(y_b\circ f)^\star}_{y_a}"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a}
\end{tikzcd}
$$

となる．そこで γ と χ を

$$
\gamma \quad \coloneqq \quad
\begin{tikzcd}
\widehat{b} \arrow[r, "(y_b)^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_{y_b\circ f}"] & a \arrow[dd, equal] \\
\widehat{b} \arrow[r, "(y_a)^\star"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_b}"] & b \arrow[r, dash, dashed] \arrow[d, "y_b"] \arrow[dr, phantom, "\Downarrow Uy_b"] & b \arrow[d, "y_b"] & \\
\widehat{b} \arrow[r, dash, dashed] & \widehat{b} \arrow[r, dash, dashed] & \widehat{b} \arrow[r, "(y_b\circ f)^\star"', "\shortmid" marking] & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
c \arrow[r, "g^\star", "\shortmid" marking] \arrow[d, "c(g{,}1)"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_b}"] & b \arrow[d, "y_b"] \\
\widehat{b} \arrow[r, dash, dashed] & \widehat{b}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
c \arrow[r, "g^\star", "\shortmid" marking] \arrow[d, "c(g{,}1)"'] \arrow[dr, phantom, "\Downarrow\sigma^{g^\star}_{y_b}"] & b \arrow[d, "y_b"] \\
\widehat{b} \arrow[r, dash, dashed] & \widehat{b}
\end{tikzcd}
$$

により定義すれば

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[d, equal] \\
c \arrow[r, "g^\star", "\shortmid"{marking}] \arrow[d, "c(g{,}1)"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star"', "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
\widehat{b} \arrow[rr, "(y_b\circ f)^\star", "\shortmid"{marking}] \arrow[d, "f^{-1}"'] \arrow[drr, phantom, "\Downarrow\sigma_{y_a}^{(y_b\circ f)^\star}"] & & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[rr, dash, dashed] & & \widehat{a}
\end{tikzcd}
$$

（γ の定義）
$=$

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[rr, dash, dashed] \arrow[d, "f"] \arrow[drr, phantom, "\Downarrow\eta_f"] & & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & b \arrow[rr, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & & a \arrow[d, equal] \\
c \arrow[r, "g^\star", "\shortmid"{marking}] \arrow[d, "c(g{,}1)"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[rr, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & & a \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_{y_b\circ f}"] & a \arrow[dd, equal] \\
\widehat{b} \arrow[r, "(y_a)^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_b}"] & b \arrow[r, dash, dashed] \arrow[d, "y_b"] \arrow[dr, phantom, "\Downarrow Uy_b"] & b \arrow[d, "y_b"] & \\
\widehat{b} \arrow[r, dash, dashed] \arrow[d, "f^{-1}"'] \arrow[drrr, phantom, "\Downarrow\varepsilon_{y_b}"] & \widehat{b} \arrow[r, dash, dashed] & \widehat{b} \arrow[r, "(y_b\circ f)^\star"', "\shortmid"{marking}] & a \arrow[d, "y_b"] \\
\widehat{a} \arrow[rrr, dash, dashed] & & & a
\end{tikzcd}
$$

（χ の定義）
$=$

$$
\begin{tikzcd}
a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, dash, dashed] \arrow[d, equal] & a \arrow[dddd, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] & a \arrow[dd, "f"] \arrow[r, phantom, "\Downarrow\eta_{y_b\circ f}"] & \\
c \arrow[r, "g^\star"', "\shortmid"{marking}] \arrow[dd, "c(g{,}1)"'] \arrow[d, phantom, "\Downarrow\sigma_{y_b}^{g^\star}"] & b \arrow[d, equal] \arrow[ur, phantom, "\Downarrow\varepsilon_f"] & & \\
 & b \arrow[r, dash, dashed] \arrow[d, "y_b"'] \arrow[dr, phantom, "\Downarrow Uy_b"] & b \arrow[d, "y_b"] & \\
\widehat{b} \arrow[r, dash, dashed] \arrow[d, "f^{-1}"'] \arrow[drrr, phantom, "\Downarrow\varepsilon_{y_b}"] & \widehat{b} \arrow[r, dash, dashed] & \widehat{b} \arrow[r, "(y_b\circ f)^\star"', "\shortmid"{marking}] & a \arrow[d, "y_b"] \\
\widehat{a} \arrow[rrr, dash, dashed] & & & \widehat{a}
\end{tikzcd}
$$

$= \quad \theta$

となる．ここで β を

$$
\beta \quad := \quad
\begin{tikzcd}
a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, "(y_b\circ f)^\star", "\shortmid"{marking}] \arrow[d, "f"] \arrow[ddr, phantom, "\Downarrow\varepsilon_{y_b\circ f}"] & \widehat{b} \arrow[dd, equal] \\
b \arrow[r, dashed, no head] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta_{y_b}"] & b \arrow[d, "y_b"] & \\
b \arrow[r, "y_b^\star"', "\shortmid"{marking}] & \widehat{b} \arrow[r, dashed, no head] & \widehat{b}
\end{tikzcd}
$$

と定義するとこの β,γ は補題 58 の条件を満たす．故に γ は nullary-r-左 cocartesian である．y_a が稠密なので命題 49 より $\sigma_{y_a}^{(y_b\circ f)^\star}$ は r-各点左 Kan 拡張である．$\sigma_{y_b}^{g^\star}$ と ε_{y_b} が cartesian だから補題 5 より χ も cartesian である．

以上を踏まえて，上で θ になることを示した合成

$$
\begin{tikzcd}
 & a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] & \text{cocartesian} \arrow[l, gray] \\
\text{cocartesian} \arrow[r, gray] & b \arrow[r, dashed, no head] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[d, equal] & \\
\text{cartesian} \arrow[r, gray] & c \arrow[r, "g^\star", "\shortmid"{marking}] \arrow[d, "c(g{,}1)"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{f^\star}"] & a \arrow[d, equal] & \text{(c)} \\
\text{nullary-r-左 cocartesian} \arrow[r, gray] & \widehat{b} \arrow[r, "y_b^\star", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] & \text{(b)} \\
\text{r-各点左 Kan 拡張} \arrow[r, gray] & \widehat{b} \arrow[rr, "(y_b\circ f)^\star", "\shortmid"{marking}] \arrow[d, "f^{-1}"'] & \Downarrow\sigma_{y_a}^{(y_b\circ f)^\star} & a \arrow[d, "y_a"] & \text{(a)} \\
 & \widehat{a} \arrow[rr, dashed, no head] & & \widehat{a} &
\end{tikzcd}
$$

を考える．補題 33 により (a) の部分の合成も r-各点左 Kan 拡張である．従って r-各点左 Kan 拡張の定義より (b) の部分は左 Kan 拡張となる．η_g が cocartesian だから補題 22 より (c) の部分も左 Kan 拡張である．よって η_f も cocartesian だから全体（即ち θ）も左 Kan 拡張である．

以上により y_a は米田構造を定める．

ここで更に $\mathbb{D}$ が nullary-右 cocartesian tabulation を持つとする．即ち補題 41 より条件 (wc) が成り立つ．このとき次の条件が成り立つことを示す．

(y5) $a \in \mathcal{C}$ と $f\colon a \to b$ が admissible で，図式

$$\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[ur, "f"] \arrow[r, "y_a"'] \arrow[ur, phantom, "\eta \Uparrow", shift right=4] & \widehat{a} \end{tikzcd}$$

が絶対左 Kan リフトであるとする．このとき $\langle g, \eta \rangle$ は f に沿った y_a の c-各点左 Kan 拡張である．

> $\because$) η を
>
> $$\begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\ b \arrow[r, "f^\star", "|"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & a \arrow[d, "y_a"] \\ \widehat{a} \arrow[r, dashed, no head] & \widehat{a} \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, dashed, no head] \arrow[d, "f"'] & a \arrow[dd, "y_a"] \\ b \arrow[d, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \\ \widehat{a} \arrow[r, dashed, no head] & \widehat{a} \end{tikzcd}$$
>
> と分解すると，命題 46 より η' は cartesian である．y_a が稠密だから η' は r-各点左 Kan 拡張である．よって命題 35 により $\langle g, \eta \rangle$ は c-各点左 Kan 拡張である．

$\square$

以上により，ある程度の条件の下で米田構造と米田埋込を同一視することができる．

命題 60. $\mathbb{D}$ が augmented virtual equipment のとき，conjoint を持つ垂直射全体を A とすると，これは次の条件を満たす．

$$f\colon a \to b,\ \ g\colon b \to c,\ \ g \in A \text{ ならば } g \circ f \in A \text{ である．}$$

証明. $f\colon a \to b$，$g\colon b \to c$，$g \in A$ とする．g の conjoint $g^\star\colon c \to b$ を取ると，仮定より制限 $\langle g^\star(\mathrm{id}, f), \beta \rangle$ が存在する．

$$\begin{tikzcd} c \arrow[r, "g^\star(\mathrm{id},f)", "|"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] \\ c \arrow[r, "g^\star", "|"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & b \arrow[d, "g"] \\ c \arrow[r, dashed, no head] & c \end{tikzcd}$$

β と ε_g が cartesian だから，補題 5 より全体も cartesian である．従って命題 4（の horizontal dual）によりこれは $g \circ f$ の conjoint を定める．即ち $g \circ f \in A$ である． $\square$

この意味で米田構造における admissible とは「conjoint が存在する」に相当する言葉だと思うことができる．ここで

$$\mathrm{id}_a \text{ が conjoint を持つ} \iff a \text{ が unital}$$

である．つまり augmented virtual equipment $\mathbb{D}$ においては，unital な a に対して米田埋込を取ることができれば，定理 59 によって $\mathcal{V}(\mathbb{D})$ の米田構造が得られる．

例 61. 例 2 の (V, V')-$\mathbb{P}$ROF を考える．まず次の図式の Ψ, F, G に対して $\Phi(-,\square) \coloneqq \Psi(F-, G\square)$ と定義すると，恒等変換 id が定める cell が cartesian になる．

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Phi", "|" marking] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{B} \arrow[r, "\Psi"', "|" marking] & \mathcal{D}
\end{tikzcd}$$

故に (V, V')-$\mathbb{P}$ROF は augmented virtual equipment である．次に

$$F\colon \mathcal{C} \to \mathcal{D} \text{ が conjoint を持つ} \iff \text{任意の } c \in \mathcal{C},\ d \in \mathcal{D} \text{ に対して } \mathcal{D}(Fc, d) \in V$$

である．実際 F の conjoint は

$$\begin{aligned}
F^\star(-,\square) &\coloneqq \mathcal{D}(F-,\square) \\
(\varepsilon_F)_{cd} &\coloneqq \mathrm{id}\colon \mathcal{D}(Fc,d) \Rightarrow \mathcal{D}(Fc,d) \\
(\eta_F)_c &\coloneqq j_{Fc}\colon I \to \mathcal{D}(Fc,Fc)
\end{aligned}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "F^\star", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_F"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \arrow[r, dashed, no head] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, dashed, no head] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\eta_F"] & \mathcal{C} \arrow[d, equal] \\
\mathcal{D} \arrow[r, "F^\star"', "|" marking] & \mathcal{C}
\end{tikzcd}$$

により与えられる．よって対象 $\mathcal{C} \in (V, V')$-$\mathbb{P}$ROF が unital とは V-豊穣圏ということである．unital な $\mathcal{C}$ に対して米田埋込 $y\colon \mathcal{C} \to \widehat{\mathcal{C}}$ を考えると，例 54 より (V, V')-$\mathbb{P}$ROF における米田埋込になる．従って定理 59 より $\mathcal{V}((V, V')\text{-}\mathbb{P}\mathrm{ROF}) = V'\text{-}\mathbf{CAT}$ の米田構造が定まる．

$V = \mathbf{Set}$ の場合，Φ の tabulation は次のように構成できる．

- まず圏 $\langle\Phi\rangle$ はココンマ圏の構成で使用した

$$\begin{tikzcd}
\mathrm{Span}(A,B) \arrow[r, shift left, "\Psi"] \arrow[rr, bend left, "\uparrow"] & \mathbf{Prof}(A,B) \arrow[l, shift left, "\Lambda"] \arrow[l, phantom, "\perp"] \arrow[r, shift left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathrm{Cospan}(A,B) \arrow[l, shift left, "\Phi"] \arrow[ll, bend left, "\downarrow"]
\end{tikzcd}$$

  を使って $\Lambda(\Phi) = A \xleftarrow{Q} \langle\Phi\rangle \xrightarrow{P} B$ により定める．
- cell

$$\begin{tikzcd}
\langle J\rangle \arrow[r, dashed, no head] \arrow[d, "P_A"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J\rangle \arrow[d, "P_B"] \\
A \arrow[r, "J"', "\shortmid" marking] & B
\end{tikzcd}$$

  は $\langle b,a,f\rangle \in \langle\Phi\rangle$ に対して $f \in J(b,a)$ から定まる写像

$$\pi_{\langle b,a,f\rangle} \colon 1 \to J(b,a)$$

  とする．

この π は cocartesian である．即ち $\langle\Phi\rangle$ は nullary-右 cocartesian tabulation だから定理 59 により $\mathbf{Set}'$-$\mathbf{CAT}$ の米田構造は条件 (y5) を満たす．

一般の V に対しては V'-$\mathbf{CAT}$ の米田構造は条件 (y5) を満たすとは限らないのであった．つまりこの場合の (V,V')-$\mathbb{P}$ROF は nullary-右 cocartesian tabulation を持たないことになる． □

7 普遍随伴

$y\colon a \to \widehat{a}$ を各点稠密な米田射，$f\colon a \to b$ を垂直射として conjoint $f^\star$ が存在するとする．このとき $\mathcal{V}(\mathbb{D})$ における左 Kan 拡張 $f^\dagger y$ とは，$\mathbb{D}$ における左 Kan 拡張

$$\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

のことであった（定理 19）．今 y が各点稠密な米田射だから，この左 Kan 拡張 $f^{\star\dagger}y$ は存在する（更にこれは各点左 Kan 拡張である）．よって augmented virtual double category

における普遍随伴とは「$f^{\star\dagger}y$ は左随伴を持つか」という問題だと考えることができる．

定理 62. $y\colon a \to \widehat{a}$ を稠密な米田射，$f\colon a \to b$ を垂直射として conjoint $y^\star, f^\star$ が存在するとする．$y^\star$ に沿った f の左 Kan 拡張 $\langle y^{\star\dagger}f, \eta\rangle$ が存在するとする．更に

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
\widehat{a} \arrow[r, "\shortmid"{marking}, "y^\star"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta_y"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[dd, "f^{\star\dagger}y"'] \arrow[ddr, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[dd, "y"] \\
& \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
$$

により β を取ったとき，合成

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
\widehat{a} \arrow[r, "\shortmid"{marking}, "y^\star"'] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
\tag{63}
$$

も左 Kan 拡張であるとする．このとき $\mathcal{V}(\mathbb{D})$ における随伴 $y^{\star\dagger}f \dashv f^{\star\dagger}y$ が成り立つ．

証明. 命題 72 により次の等式を満たす κ を取る．

$$
\begin{tikzcd}
\widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, dashed, no head] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dashed, no head] \arrow[d, "f^{\star\dagger}y"'] \arrow[drr, phantom, "\Downarrow\sigma_y^{f^\star}"] & b \arrow[r, "\shortmid"{marking}, "f^\star"'] & a \arrow[d, "y"] \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\widehat{a} \arrow[r, dashed, no head] \arrow[d, "y^{\star\dagger}f"'] \arrow[ddr, phantom, "\Downarrow\kappa"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon_y"] & a \arrow[dd, "y"] \\
b \arrow[d, "f^{\star\dagger}y"'] & & \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
$$

更に (63) が左 Kan 拡張なので，次の cell ε が取れる．

$$
\begin{tikzcd}
b \arrow[r, dashed, no head] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
& \widehat{a} \arrow[r, "\shortmid"{marking}, "y^\star"'] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\
b \arrow[r, dashed, no head] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[dd, "f"] \\
& \\
b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

この κ と ε が $y^{\star\dagger}f \dashv f^{\star\dagger}y$ を与えることを示せばよい．まず

$$
\begin{tikzcd}
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "f^{\star\dagger}y"] & a \arrow[ddd, "y"] \\
{} \arrow[r, phantom, "\Downarrow\varepsilon"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] & \widehat{a} \arrow[dd, equal] \arrow[r, phantom, "\Downarrow\sigma_y^{f^\star}"] & {} \\
b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[d, "f^{\star\dagger}y"] \arrow[r, phantom, "\Downarrow\kappa"] & {} & {} \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "f^{\star\dagger}y"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, equal] \\
{} \arrow[r, phantom, "\Downarrow\varepsilon"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] & \widehat{a} \arrow[dd, equal] \arrow[r, "y^\star"', "\shortmid"{marking}] \arrow[ddr, phantom, "\Downarrow\varepsilon_y"] & a \arrow[dd, "y"] \\
b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[d, "f^{\star\dagger}y"] \arrow[r, phantom, "\Downarrow\kappa"] & {} & {} \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a}
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
{} \arrow[r, phantom, "\Downarrow\varepsilon"] & \widehat{a} \arrow[r, "y^\star"', "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[rr, dash, dashed] & & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\varepsilon_f"] & a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
{} & {} & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[rr, dash, dashed] & & \widehat{a}
\end{tikzcd}
$$

$$
= \quad \sigma_y^{f^\star}
$$

となるから，左 Kan 拡張 $\sigma_y^{f^\star}$ の普遍性により

$$
\begin{tikzcd}
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[d, "f^{\star\dagger}y"] \\
{} \arrow[r, phantom, "\Downarrow\varepsilon"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] & \widehat{a} \arrow[dd, equal] \\
b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow Uf^{\star\dagger}y"] & b \arrow[d, "f^{\star\dagger}y"] \arrow[r, phantom, "\Downarrow\kappa"] & {} \\
\widehat{a} \arrow[r, dash, dashed] & \widehat{a} \arrow[r, dash, dashed] & \widehat{a}
\end{tikzcd}
\quad = \quad \mathrm{id}
$$

となる．同様にして

$$
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\kappa"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\eta"] & a \arrow[ddd, "f"] \\
b \arrow[r, dash, dashed] \arrow[ddd, equal, start anchor=south, end anchor=north] \arrow[dr, phantom, "\Downarrow\varepsilon"'] & b \arrow[d, "f^{\star\dagger}y"'] & & \\
 & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[d, "y^{\star\dagger}f"] & \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
=
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\kappa"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] \arrow[dd, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_y"] & a \arrow[r, dash, dashed] \arrow[dd, "y"] \arrow[dr, phantom, "\Downarrow\eta_y"] & a \arrow[dd, equal] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"'] & b \arrow[d, "f^{\star\dagger}y"'] & & & \\
 & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\eta"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b \arrow[rr, dash, dashed] & & b
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[ddr, phantom, "\Downarrow\eta_y"] & a \arrow[dd, equal] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"'] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[drr, phantom, "\Downarrow\sigma_y^{f^\star}"] & b \arrow[r, "f^\star"', "\shortmid"{marking}] & a \arrow[d, "y"] & \\
 & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"] \arrow[drr, phantom, "\Downarrow\eta"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b \arrow[rr, dash, dashed] & & b
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, dash, dashed] \arrow[d, equal] \arrow[dddr, phantom, "\Downarrow\eta_y"] & a \arrow[ddd, equal] \\
b \arrow[r, dash, dashed] \arrow[dddd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"'] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[drr, phantom, "\Downarrow\beta"] & b \arrow[r, "f^\star"', "\shortmid"{marking}] & a \arrow[d, equal] & \\
 & \widehat{a} \arrow[rr, "y^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\varepsilon_y"] & & a \arrow[d, "y"] & \\
 & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"] \arrow[drr, phantom, "\Downarrow\eta"] & \widehat{a} \arrow[r, "y^\star", "\shortmid"{marking}] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b \arrow[rr, dash, dashed] & & b
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, "y^{\star}", "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[r, dash, dashed] \arrow[d, "f^{\star\dagger}y"'] \arrow[drr, phantom, "\Downarrow\beta"] & b \arrow[r, "f^{\star}"', "\shortmid"{marking}] & a \arrow[d, equal] \\
 & \widehat{a} \arrow[rr, "y^{\star}"', "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"] \arrow[drr, phantom, "\Downarrow\eta"] & & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[rr, dash, dashed] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, "y^{\star}", "\shortmid"{marking}] \arrow[d, "y^{\star\dagger}f"] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, dash, dashed] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, dash, dashed] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\varepsilon_f"] & b \arrow[r, dash, dashed] & b \arrow[r, "f^{\star}"', "\shortmid"{marking}] & a \arrow[d, "f"] \\
b \arrow[rrr, dash, dashed] & & & b
\end{tikzcd}
$$

$$
= \quad \eta
$$

となるから

$$
\begin{tikzcd}
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"] \arrow[ddr, phantom, "\Downarrow\kappa"] & \widehat{a} \arrow[dd, equal] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[d, "f^{\star\dagger}y"'] & \\
 & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "y^{\star\dagger}f"'] \arrow[dr, phantom, "\Downarrow Uy^{\star\dagger}f"] & \widehat{a} \arrow[d, "y^{\star\dagger}f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
\quad = \quad \mathrm{id}
$$

も分かる. □

系 64. r-各点左 Kan 拡張 $y^{\star\dagger}f$ が存在するならば $y^{\star\dagger}f \dashv f^{\star\dagger}y$ である. □

補題 65. $f \dashv u\colon a \to b$ を $\mathcal{V}(\mathbb{D})$ における随伴として, counit を ε とする. このとき $\mathbb{D}$ における次の図式において

$$
\beta \text{ が cartesian である} \iff \text{全体が cartesian である.}
$$

$$
\begin{tikzcd}
c \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & c \arrow[r, "\vec{J}", "\|"{marking}] \arrow[d, "h"] \arrow[ddr, phantom, "\Downarrow\beta"] & d \arrow[dd, "k"] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon"] & b \arrow[d, "u"'] & \\
 & a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

証明. 次の右辺に現れる cell θ に対して $\widetilde{\theta}$ を

$$
\begin{tikzcd}
u \arrow[r, "\vec{K}", "\shortmid"{marking}] \arrow[d, "j"'] & v \arrow[d, "l"] \\
c \arrow[ddd, "h"'] \arrow[r, phantom, "\Downarrow\widetilde{\theta}"] & d \arrow[dd, "k"] \\
 & \\
 & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
\quad \coloneqq \quad
\begin{tikzcd}
u \arrow[r, dash, dashed] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow Uj"] & u \arrow[r, "\vec{K}", "\shortmid"{marking}] \arrow[d, "j"] & v \arrow[d, "l"] \\
c \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & c \arrow[d, "h"] \arrow[r, phantom, "\Downarrow\theta"] & d \arrow[dd, "k"] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[d, "u"] & \\
\arrow[r, phantom, "\Downarrow\varepsilon"] & a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

と定義すると，これにより θ と $\widetilde{\theta}$ は 1 対 1 に対応する．よって

$$
\begin{tikzcd}
u \arrow[r, "\vec{K}", "\shortmid"{marking}] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\tau"] & v \arrow[d, "l"] \\
c \arrow[r, "\vec{J}"', "\shortmid"{marking}] \arrow[d, "h"'] & d \arrow[dd, "k"] \\
b \arrow[d, "u"'] \arrow[r, phantom, "\Downarrow\beta"] & \\
a \arrow[r, dash, dashed] & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[r, "\vec{K}", "\shortmid"{marking}] \arrow[d, "j"'] & v \arrow[d, "l"] \\
c \arrow[d, "h"'] \arrow[r, phantom, "\Downarrow\theta"] & d \arrow[dd, "k"] \\
b \arrow[d, "u"'] & \\
a \arrow[r, dash, dashed] & a
\end{tikzcd}
$$

$$
\iff \quad
\begin{tikzcd}
u \arrow[rr, "\vec{K}", "\shortmid"{marking}] \arrow[d, "j"'] \arrow[drr, phantom, "\Downarrow\tau"] & & v \arrow[d, "l"] \\
c \arrow[r, dash, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & c \arrow[r, "\vec{J}", "\shortmid"{marking}] \arrow[d, "h"] \arrow[r, phantom, shift right=8, "\Downarrow\beta"] & d \arrow[dd, "k"] \\
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[d, "u"'] & \\
\arrow[r, phantom, "\Downarrow\varepsilon"] & a \arrow[r, dash, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u \arrow[r, "\vec{K}", "\shortmid"{marking}] \arrow[d, "j"'] & v \arrow[d, "l"] \\
c \arrow[ddd, "h"'] \arrow[r, phantom, "\Downarrow\widetilde{\theta}"] & d \arrow[dd, "k"] \\
 & \\
 & a \arrow[d, "f"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

となるから補題が成り立つ． $\square$

定理 66. $y\colon a \to \widehat{a}$ を稠密な米田射，$f\colon a \to b$ を垂直射として conjoint $f^\star$ が存在するとする．$\mathcal{V}(\mathbb{D})$ における随伴 $l \dashv f^{\star\dagger}y$ が存在するならば，任意の $J\colon u \nrightarrow a$ に沿った f の r-各点左 Kan 拡張が存在する．

証明. $\mathcal{V}(\mathbb{D})$ における随伴 $l \dashv f^{\star\dagger}y$ が成り立つとして，その counit を ε とする．合成

$$
\begin{tikzcd}
b \arrow[r, dash, dashed] \arrow[dd, equal] & b \arrow[r, "f^\star", "|" marking] \arrow[d, "f^{\star\dagger}y"'] \arrow[dr, phantom, "\Downarrow\sigma_y^{f^\star}"] & a \arrow[d, "y"] \\
\Downarrow\varepsilon & \widehat{a} \arrow[r, dash, dashed] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & \widehat{a} \arrow[d, "l"] \\
b \arrow[r, dash, dashed] & b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

は cartesian である（補題 65）．即ち $f^\star$ は $l \circ y$ の conjoint を与える．よって普遍性から $l \circ y \cong f$ が分かる．このとき命題 37 より

$$
\begin{tikzcd}
u \arrow[r, "J", "|" marking] \arrow[d, "J^\dagger y"'] \arrow[dr, phantom, "\Downarrow\sigma_y^J"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dash, dashed] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & \widehat{a} \arrow[d, "l"] \\
b \arrow[r, dash, dashed] & b
\end{tikzcd}
$$

が r-各点左 Kan 拡張になる． □

定理 62 では $y^\star$ を使ったが，定理 66 では $y^\star$ は関係ないことを注意しておく．そこで augmented virtual double category における total を次のように定義する．

定義. $f\colon a \to b$ が **total**
$\iff$ 任意の $J\colon u \nrightarrow a$ に沿った f の r-各点左 Kan 拡張が存在する．

系 67. $y\colon a \to \widehat{a}$ を稠密な米田射，$f\colon a \to b$ を垂直射として conjoint $y^\star, f^\star$ が存在するとき

$$f \text{ が total である} \iff f^{\star\dagger}y \text{ は } \mathcal{V}(\mathbb{D}) \text{ における左随伴を持つ．}$$ □

定義. 対象 a が **total** $\iff$ id_a が total である．

命題 68. 米田埋込 $y\colon a \to \widehat{a}$ が $\mathcal{V}(\mathbb{D})$ において左随伴を持つならば，a は total である．

証明. $\mathcal{V}(\mathbb{D})$ における随伴 $l \dashv y$ が成り立つとして，その counit を ε とする．y は忠実充満だから，命題 48 により $\mathcal{V}(\mathbb{D})$ において忠実充満である．よって ε は同型である．そこで $J\colon u \nrightarrow a$ に対して合成

$$\begin{tikzcd}
u \arrow[r, "J", "|" marking] \arrow[d, "J^\dagger y"'] \arrow[dr, phantom, "\Downarrow\sigma^J_y"] & a \arrow[r, dashed, no head] \arrow[d, "y"] & a \arrow[dd, equal] \\
\widehat{a} \arrow[r, dashed, no head] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & \widehat{a} \arrow[d, "l"] \arrow[r, phantom, "\Downarrow\varepsilon^{-1}"] & {} \\
a \arrow[r, dashed, no head] & a \arrow[r, dashed, no head] & a
\end{tikzcd}$$

を考えると，σ^J_y が r-各点左 Kan 拡張で l が左随伴だから全体も r-各点左 Kan 拡張である．即ち a は total である． □

命題 69. 米田埋込 $y\colon a \to \widehat{a}$ の conjoint $y^\star$ が存在して，更に a は unital であるとする．このとき a が total ならば，y は $\mathcal{V}(\mathbb{D})$ において左随伴を持つ．

証明. $f \coloneqq \mathrm{id}_a$ として定理 62 を使う．まず a が unital だから id_a の conjoint I_a が存在する．次に a が total だから，r-各点左 Kan 拡張 $y^{\star\dagger}\mathrm{id}_a$ が存在する．よって定理 62 が使えて更に $y^{\star\dagger}\mathrm{id}_a \dashv I_a^\dagger y \cong y$ となる． □

系 70. augmented virtual equipment $\mathbb{D}$ が次の条件 (uyr) を満たすとする．

(uyr) unital な a に対して米田埋込 $y\colon a \to \widehat{a}$ が与えられ，更に任意の $f\colon b \to \widehat{a}$ に対して制限 $\widehat{a}(f, y)$ が存在する．

命題 60 の直後の注意により $\mathcal{V}(\mathbb{D})$ の米田構造が得られる．このとき

(1) unital な $a \in \mathbb{D}$ に対して

a が米田構造の意味で total である $\Longleftrightarrow$ a が $\mathbb{D}$ において total である．

(2) a が unital で $f\colon a \to b$ が conjoint を持つとき

f が米田構造の意味で total である $\Longleftrightarrow$ f が $\mathbb{D}$ において total で，$f^{\star\dagger}y$ の左随伴が conjoint を持つ．

□

定義. 次のような cell と垂直射 $g\colon b_n \to c$ を考える.

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortparallel" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[d, "f_1"] \\
b_0 \arrow[r, "\vec{K}_1"', "\shortparallel" marking] & b_1
\end{tikzcd}
,\quad \cdots ,\quad
\begin{tikzcd}
a_{n-1} \arrow[r, "\vec{J}_n", "\shortparallel" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_n \arrow[d, "f_n"] \\
b_{n-1} \arrow[r, "\vec{K}_n"', "\shortparallel" marking] & b_n
\end{tikzcd}
$$

このとき $\langle\beta_1, \cdots, \beta_n\rangle$ が**左 g-完全** $\iff$ 図式

$$
\begin{tikzcd}
a_0 \arrow[r, "\vec{J}_1", "\shortparallel" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\beta_1"] & a_1 \arrow[r, "\vec{J}_2", "\shortparallel" marking] \arrow[d, "f_1"] & \cdots \arrow[r, "\vec{J}_{n-1}", "\shortparallel" marking] & a_{n-1} \arrow[r, "\vec{J}_n", "\shortparallel" marking] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\Downarrow\beta_n"] & a_n \arrow[d, "f_n"] \\
b_0 \arrow[r, "\vec{K}_1"', "\shortparallel" marking] \arrow[d, "l"'] & b_1 \arrow[r, "\vec{K}_2"', "\shortparallel" marking] & \cdots \arrow[r, "\vec{K}_{n-1}"', "\shortparallel" marking] \arrow[d, phantom, "\Downarrow\eta"] & b_{n-1} \arrow[r, "\vec{K}_n"', "\shortparallel" marking] & b_n \arrow[d, "g"] \\
c \arrow[rrrr, dashed, no head] & & & & c
\end{tikzcd}
$$

において η が左 Kan 拡張ならば全体も左 Kan 拡張である.

定義. $\langle\beta_1, \cdots, \beta_n\rangle$ が**左完全** $\iff$ 任意の g に対して $\langle\beta_1, \cdots, \beta_n\rangle$ が左 g-完全である.

またこれらの定義の中の左 Kan 拡張を r-各点左 Kan 拡張に変えたものを **r-左 g-完全**, **r-左完全**という.

補題 71. nullary-r-左 cocartesian パスは r-左完全である.

証明. 補題 33 より明らか. □

命題 72. $y\colon a \to \widehat{a}$ を各点稠密な米田射とするとき, 次の等式により cell θ と θ' が 1 対 1 に対応する.

$$
\begin{tikzcd}
u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "f"'] & v \arrow[r, "J", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\theta"] & a \arrow[d, equal] \\
w \arrow[rr, "\vec{K}", "\shortparallel" marking] \arrow[d, "\vec{K}^\dagger y"'] & \arrow[d, phantom, "\Downarrow\sigma_y^{\vec{K}}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "f"'] & v \arrow[r, "J", "\shortmid" marking] \arrow[dd, "J^\dagger y"] \arrow[dr, phantom, "\Downarrow\sigma_y^J"] & a \arrow[dd, "y"] \\
w \arrow[d, "\vec{K}^\dagger y"'] \arrow[r, phantom, "\Downarrow\theta'"] & & \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
$$

証明. $\sigma_y^{\vec{K}}$ が cartesian であることと, σ_y^J が各点左 Kan 拡張であることから分かる. □

命題 73. 各点稠密な米田射 $y_a\colon a \to \widehat{a}$ に対して命題 72 で対応する θ と θ' を取るとき

$$\theta \text{ が r-左 } y\text{-完全である} \iff \theta' \text{ が r-各点左 Kan 拡張である.}$$

$$\begin{tikzcd}
u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "f"'] & v \arrow[r, "J", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\theta"] & a \arrow[d, equal] \\
w \arrow[rr, "\vec{K}", "\shortmid" marking] \arrow[d, "\vec{K}^{\dagger}y"'] & \arrow[d, phantom, "\Downarrow\sigma_y^{\vec{K}}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "\vec{H}", "\shortparallel" marking] \arrow[d, "f"'] & v \arrow[r, "J", "\shortmid" marking] \arrow[dd, "J^{\dagger}y"] \arrow[dr, phantom, "\Downarrow\sigma_y^J"] & a \arrow[dd, "y"] \\
w \arrow[d, "y\vec{K}y"'] \arrow[r, phantom, "\Downarrow\theta'"] & & \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

証明. 命題 32 から明らか. □

命題 74. $\mathbb{D}$ を augmented virtual equipment として，$y\colon a \to \widehat{a}$ を各点稠密な米田射とする．y が conjoint を持つならば $\widehat{a}$ は total である．

証明. 水平射 $J\colon u \nrightarrow \widehat{a}$ に対して r-各点左 Kan 拡張

$$\begin{tikzcd}
u \arrow[r, "J", "\shortmid" marking] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\eta"] & \widehat{a} \arrow[d, equal] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

が存在することを示せばよい．まず仮定より制限 $\langle J(\mathrm{id}, y), \beta\rangle$ が存在するので，補題 12（の証明）より合成 $\langle J \odot y^{\star}, \gamma\rangle$ も存在し，それは

$$\begin{tikzcd}
u \arrow[r, "J", "\shortmid" marking] \arrow[d, equal] & \widehat{a} \arrow[r, "y^{\star}", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
u \arrow[rr, "J(\mathrm{id},y) = J\odot y^{\star}", "\shortmid" marking] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\beta"] & a \arrow[d, "y"] \\
u \arrow[rr, "J"', "\shortmid" marking] & & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "J", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & \widehat{a} \arrow[r, "y^{\star}", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta_y"] & a \arrow[d, "y"] \\
u \arrow[r, "J"', "\shortmid" marking] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

により得られる．補題 58 により γ は nullary-r-左 cocartesian である．よって補題 71 より γ は r-左 exact となる．そこで命題 72 を使って γ に対応する γ' を取ると

$$\begin{tikzcd}
u \arrow[r, "J", "\shortmid" marking] \arrow[d, equal] & \widehat{a} \arrow[r, "y^{\star}", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
u \arrow[rr, "J(\mathrm{id},y)", "\shortmid" marking] \arrow[d, "J(\mathrm{id},y)^{\dagger}y"'] & \arrow[d, phantom, "\Downarrow\sigma_y^{J(\mathrm{id},y)}"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "J", "\shortmid" marking] \arrow[d, "J(\mathrm{id},y)^{\dagger}y"'] \arrow[dr, phantom, "\Downarrow\gamma'"] & \widehat{a} \arrow[r, "y^{\star}", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta_y"] & a \arrow[d, "y"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}$$

命題 73 により γ' が r-各点左 Kan 拡張になる． □

※ この定理と同様のものを米田構造に対しても示している（「米田構造」の PDF を参照）．但しそのときは a に small という条件が付いていた．即ち $\widehat{a}$ が admissible でなければ $\widehat{a}$ が total とは言えなかった．一般に V-豊穣圏が小とは限らなくても，$\widehat{\mathcal{C}}$ の米田埋込 $y\colon \widehat{\mathcal{C}} \to \widehat{\widehat{\mathcal{C}}}$ は V'-**CAT** において左随伴を持つことが分かる（「Universe Enlargement」の PDF を参照）．よって augmented virtual double category の方がより良いものになっていると言える．

定理 75. a を total として $f\colon a \to b$ の conjoint $f^\star$ が存在するとする．このとき

$$f \text{ が } \mathcal{V}(\mathbb{D}) \text{ において右随伴を持つ} \iff f \text{ が任意の左 Kan 拡張と交換する．}$$

証明. ($\Longrightarrow$) 命題 23 から明らか.

($\Longleftarrow$) a が total だから左 Kan 拡張 $(f^\star)^\dagger\mathrm{id}_a$ が存在する．これが f と交換するから，命題 25 により $f \dashv f^{\star\dagger}\mathrm{id}_a$ である． □

以下では $f\colon a \to b$ を垂直射，$y_a\colon a \to \widehat{a}$ と $y_b\colon b \to \widehat{b}$ を稠密な米田射として conjoint $(y_b \circ f)^\star$ が存在するとしておく．このとき $f^{-1} \coloneqq (y_b \circ f)^{\star\dagger}y_a\colon \widehat{b} \to \widehat{a}$ が定義できる．

※ $(y_b \circ f)^\star \cong y_b^\star(f, \mathrm{id})$ だから，$\mathbb{D}$ が左制限を持つ場合は y_b が conjoint を持てばこの条件は満たされる．

命題 76. $J\colon c \nrightarrow b$ を水平射として，conjoint $f^\star, y_b^\star$ と制限 $\langle \widehat{b}(J^\dagger y_b, y_b \circ f), \theta \rangle$ が存在するとする．このとき $f^{-1} \circ (J^\dagger y_b) \cong J(\mathrm{id}, f)^\dagger y_b$ である．

証明. まず記号を準備する．左 Kan 拡張 $\sigma^J_{y_b}$ を

$$\begin{tikzcd}
c \arrow[r, "J", "|" marking] \arrow[d, "J^\dagger y_b"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_b}"] & b \arrow[d, "y_b"] \\
\widehat{b} \arrow[r, dashed, no head] & \widehat{b}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
c \arrow[r, "J", "|" marking] \arrow[dd, "J^\dagger y_b"'] \arrow[ddr, phantom, "\Downarrow\sigma^J_{y_b}"] & b \arrow[dd, "y_b"] \\
& \\
\widehat{b} \arrow[r, dashed, no head] & \widehat{b}
\end{tikzcd}$$

と分解する．補題 5 より χ は cartesian である．

次に conjoint $(y_b \circ f)^\star$ と制限 $\langle \widehat{b}(J^\dagger y_b, y_b \circ f), \theta \rangle$ が存在するから

$$
\begin{tikzcd}
c \arrow[r, "|" marking, "\widehat{b}(J^\dagger y_b, y_b \circ f)"] \arrow[d, "J^\dagger y_b"'] \arrow[dr, phantom, "\Downarrow\theta'"] & a \arrow[d, equal] & & c \arrow[r, "|" marking, "\widehat{b}(J^\dagger y_b, y_b \circ f)"] \arrow[d, "J^\dagger y_b"'] & a \arrow[d, equal] \\
\widehat{b} \arrow[r, "|" marking, "(y_b \circ f)^\star"'] \arrow[d, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon_{y_b \circ f}"] & a \arrow[d, "f"] & = & \widehat{b} \arrow[d, equal] \arrow[r, phantom, "\Downarrow\theta"] & a \arrow[d, "f"] \\
\widehat{b} \arrow[d, equal] & b \arrow[d, "y_b"] & & \widehat{b} \arrow[d, equal] & b \arrow[d, "y_b"] \\
\widehat{b} \arrow[r, dashed, no head] & \widehat{b} & & \widehat{b} \arrow[r, dashed, no head] & \widehat{b}
\end{tikzcd}
$$

と分解すれば θ' も cartesian である．

更に conjoint $y_b^\star$ が存在するから

$$
\begin{tikzcd}
\widehat{b} \arrow[r, "|" marking, "(y_b \circ f)^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] & & \widehat{b} \arrow[r, "|" marking, "(y_b \circ f)^\star"] \arrow[dd, equal] & a \arrow[d, "f"] \\
\widehat{b} \arrow[r, "|" marking, "y_b^\star"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_{y_b}"] & b \arrow[d, "y_b"] & = & \arrow[r, phantom, "\Downarrow\varepsilon_{y_b \circ f}"] & b \arrow[d, "y_b"] \\
\widehat{b} \arrow[r, dashed, no head] & \widehat{b} & & \widehat{b} \arrow[r, dashed, no head] & \widehat{b}
\end{tikzcd}
$$

と分解する．この β も cartesian だから，これは制限 $y_b^\star(\mathrm{id}, f)$ を与える．よって補題 12 により合成 $y_b^\star \odot f^\star$ も存在する．これは次の γ により与えられる．

$$
\begin{tikzcd}
\widehat{b} \arrow[r, "|" marking, "y_b^\star"] \arrow[d, equal] & b \arrow[r, "|" marking, "f^\star"] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] & & \widehat{b} \arrow[r, "|" marking, "y_b^\star"] \arrow[dd, equal] & b \arrow[r, "|" marking, "f^\star"] \arrow[dd, equal] & a \arrow[dd, "f"] \\
\widehat{b} \arrow[rr, "|" marking, "(y_b \circ f)^\star"'] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\beta"] & a \arrow[d, "f"] & = & \arrow[r, phantom, "\Downarrow\mathrm{id}_{y_b^\star}"] & \arrow[r, phantom, "\Downarrow\varepsilon_f"] & \\
\widehat{b} \arrow[rr, "|" marking, "y_b^\star"'] & & b & & \widehat{b} \arrow[r, "|" marking, "y_b^\star"'] & b \arrow[r, dashed, no head] & b
\end{tikzcd}
$$

補題 58 よりこの γ は nullary-r-左 cocartesian である．

以上を踏まえて合成

$$
\begin{tikzcd}
c \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "J^\dagger y_b"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
\widehat{b} \arrow[rr, "(y_b\circ f)^\star"', "\shortmid"{marking}] \arrow[d, "f^{-1}"'] \arrow[drr, phantom, "\Downarrow\sigma_{y_a}^{(y_b\circ f)^\star}"] & & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[rr, dashed] & & \widehat{a}
\end{tikzcd}
$$

を考える．y_a が稠密だから，命題 49 より $\sigma_{y_a}^{(y_b\circ f)^\star}$ は r-各点左 Kan 拡張である．よって補題 33 により (a) の部分も r-各点左 Kan 拡張である．次に nullary-r-左 cocartesian の定義より，(b) の部分を

$$
\begin{tikzcd}
c \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma'"] & a \arrow[d, equal] \\
c \arrow[rr, "\widehat{b}(J^\dagger y_b, y_b\circ f)"', "\shortmid"{marking}] \arrow[d, "J^\dagger y_b"'] \arrow[drr, phantom, "\Downarrow\theta'"] & & a \arrow[d, equal] \\
\widehat{b} \arrow[rr, "(y_b\circ f)^\star"', "\shortmid"{marking}] & & a
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "J^\dagger y_b"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star"', "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
\widehat{b} \arrow[rr, "(y_b\circ f)^\star"', "\shortmid"{marking}] & & a
\end{tikzcd}
$$

と分解して γ' を定めると，γ' は nullary-左 cocartesian である．即ち

$$
\begin{tikzcd}
c \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma'"] & a \arrow[d, equal] \\
c \arrow[rr, "\widehat{b}(J^\dagger y_b, y_b\circ f)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\beta_2"] & & a \arrow[d, "f"] \\
c \arrow[rr, "J"', "\shortmid"{marking}] & & b
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\mathrm{id}_J"] & b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[dd, "f"] \\
 & & \\
c \arrow[r, "J"', "\shortmid"{marking}] & b \arrow[r, dashed] & b
\end{tikzcd}
$$

により β_2 を定めると，補題 14（の直後の注）により β_2 は cartesian である．即ちこれは

制限 $J(\mathrm{id}, f)$ を定める．このとき (c) を γ' を使って

$$
\begin{tikzcd}
c \arrow[r, "J", "\shortmid" marking] \arrow[d, equal] & b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\gamma'"] & a \arrow[d, equal] \\
\widehat{b} \arrow[rr, "J(\mathrm{id},f)"', "\shortmid" marking] \arrow[d, "J^\dagger y_b"'] & & a \arrow[dd, "y_a"] \\
\widehat{b} \arrow[d, "f^{-1}"'] & \Downarrow\varphi & \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "J", "\shortmid" marking] \arrow[d, "J^\dagger y_b"'] \arrow[dr, phantom, "\Downarrow\chi"] & b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
\widehat{b} \arrow[r, "y_b^\star", "\shortmid" marking] \arrow[d, equal] & b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\gamma"] & a \arrow[d, equal] \\
\widehat{b} \arrow[rr, "(y_b\circ f)^\star"', "\shortmid" marking] \arrow[d, "f^{-1}"'] & \Downarrow\sigma_{y_a}^{(y_b\circ f)^\star} & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
$$

と分解すれば補題 22 により φ は左 Kan 拡張である．即ち $f^{-1} \circ (J^\dagger y_b) \cong J(\mathrm{id}, f)^\dagger y_b$ である． □

命題 77. $y_a \colon a \to \widehat{a}$ は各点稠密であるとして，更に conjoint $f^\star, y_b^\star, (f^{\star\dagger} y_a)^\star$ と制限 $\widehat{b}(y_b \circ f, (f^{\star\dagger} y_a)^{\star\dagger} y_b)$ も存在するとする．このとき $f^{-1} \dashv (f^{\star\dagger} y_a)^{\star\dagger} y_b$ となる．

証明. $J \coloneqq (f^{\star\dagger} y_a)^\star$ に対して命題 76（の証明）を適用することで，左 Kan 拡張 η（次の図式の右辺）が得られる．y_a が各点稠密なのでこれは r⁺-各点左 Kan 拡張である．この η を各点左 Kan 拡張 $f^{\star\dagger} y_a$ の普遍性により左辺のように分解する．

$$
\begin{tikzcd}
\widehat{b} \arrow[r, "y_b^\star", "\shortmid" marking] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, "f^{\star\dagger} y_a"'] \arrow[dr, phantom, "\Downarrow\sigma_{y_a}^{f^\star}"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[r, dashed, no head] & \widehat{a} \arrow[r, dashed, no head] & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
\widehat{b} \arrow[r, "y_b^\star", "\shortmid" marking] \arrow[d, "f^{-1}"'] & b \arrow[r, "f^\star", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\eta"] & a \arrow[d, "y_a"] \\
\widehat{a} \arrow[rr, dashed, no head] & & \widehat{a}
\end{tikzcd}
$$

このとき命題 38 により τ も r⁺-各点左 Kan 拡張である．即ち $y_b^{\star\dagger}(f^{\star\dagger} y_a) \cong f^{-1}$ となる．よって系 64 により $f^{-1} \dashv (f^{\star\dagger} y_a)^{\star\dagger} y_b$ である． □

命題 78. conjoint $y_a^\star$ と r-各点左 Kan 拡張 $\mathrm{lan}_f \coloneqq y_a^{\star\dagger}(y_b \circ f)$ も存在するとする．このとき $\mathrm{lan}_f \dashv f^{-1}$ である．

証明. 系 64, 67 より明らか． □

8 各種 double category の関係

この節では pseudo double category，virtual double category，augmented virtual double category にどのような関係があるかを事実だけ述べる．詳細は [1] や [2] を参照．

まず augmented virtual double category $\mathbb{D}$ に対して $U(\mathbb{D})$ を以下のように定めると virtual double category になる．

- $U(\mathbb{D})$ の対象，垂直射，水平射は $\mathbb{D}$ と同じとする．
- $U(\mathbb{D})$ の cell は，$\mathbb{D}$ における次の形の cell とする．

$$\begin{tikzcd} a \arrow[r, "J_1", "\shortmid"{marking}] \arrow[d, "f"'] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] & \cdots \arrow[r, "J_{n-1}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\beta"] & a_{n-1} \arrow[r, "J_n", "\shortmid"{marking}] & u \arrow[d, "g"] \\ b \arrow[rrrr, "K"', "\shortmid"{marking}] & & & & v \end{tikzcd}$$

- 合成は $\mathbb{D}$ での合成を使う．

命題 79. $\mathbb{D} \mapsto U(\mathbb{D})$ は strict 2-functor $U\colon \mathbf{AVDCat} \to \mathbf{VDCat}$ を定める．（但し **AVDCat** は augmented virtual double category がなす strict 2-category，**VDCat** は virtual double category がなす strict 2-category である．定義は [2] を参照.） □

unital な augmented virtual double category がなす充満部分 2-category を $\mathbf{AVDCat}_u \subset \mathbf{AVDCat}$ と書く．また unital な virtual double category と normal functor がなす局所充満部分 2-category を $\mathbf{VDCat}_u \subset \mathbf{VDCat}$ と書く．

定理 80. U は **CAT**-同値 $U\colon \mathbf{AVDCat}_u \to \mathbf{VDCat}_u$ を与える． □

これにより，horizontal unit がある場合は augmented virtual double category と virtual double category は同じものだと言ってよい．

次に pseudo double category については次のようになる．

命題 81. augmented virtual double category $\mathbb{D}$ の水平射 $a \overset{J}{\nrightarrow} b \overset{K}{\nrightarrow} c \overset{H}{\nrightarrow} d$ に対して，合成 $J \odot K$ が存在するとき

$$(J \odot K) \odot H \text{ が存在する} \iff J \odot K \odot H \text{ が存在する}.$$

証明. 合成 $J \odot K$ を与える cocartesian cell を γ とすると等式

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\gamma"] & c \arrow[r, "H", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & d \arrow[d, equal] \\
a \arrow[rr, "J\odot K"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\theta'"] & c \arrow[r, "H"', "\shortmid"{marking}] & d \arrow[d, equal] \\
a \arrow[rrr, "L"', "\shortmid"{marking}] & & & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, equal] & b \arrow[r, "K", "\shortmid"{marking}] & c \arrow[r, "H", "\shortmid"{marking}] & d \arrow[dd, equal] \\
& \arrow[r, phantom, "\Downarrow\theta"] & & \\
a \arrow[rrr, "L"', "\shortmid"{marking}] & & & d
\end{tikzcd}
$$

において補題 11（の直後）を使えば

$$\theta' \text{ が cocartesian} \iff \theta \text{ が cocartesian}$$

である. □

cocartesian cell の普遍性により，この場合 $(J \odot K) \odot H \cong J \odot K \odot H$ である．また同様にして (合成が存在するならば) $J \odot (K \odot H) \cong J \odot K \odot H$ も言えるから

$$(J \odot K) \odot H \cong J \odot (K \odot H)$$

が分かる．

pseudo double category $\mathbb{D}$ に対して $W(\mathbb{D})$ を以下のように定めると virtual double category になる．

- $W(\mathbb{D})$ の対象，垂直射，水平射は $\mathbb{D}$ と同じとする．
- $W(\mathbb{D})$ の cell

$$
\begin{tikzcd}
a \arrow[r, "J_1", "\shortmid"{marking}] \arrow[d, "f"'] & a_1 \arrow[r, "J_2", "\shortmid"{marking}] & \cdots \arrow[r, "J_{n-1}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\beta"] & a_{n-1} \arrow[r, "J_n", "\shortmid"{marking}] & u \arrow[d, "g"] \\
b \arrow[rrrr, "K"', "\shortmid"{marking}] & & & & v
\end{tikzcd}
$$

は，$\mathbb{D}$ における次の形の cell とする．

$$
\begin{tikzcd}
a \arrow[r, "(\cdots(J_1\odot J_2)\odot\cdots)\odot J_n", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
$$

- 合成は $\mathbb{D}$ での合成を使う．

命題 82. virtual double category $\mathbb{D}$ に対して
ある pseudo double category $\mathbb{E}$ により $\mathbb{D} \cong W(\mathbb{E})$ となる $\iff$ $\mathbb{D}$ は合成可能かつ unital である. $\square$

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