モナド

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この PDF では $\mathcal{C}$ を strict 2-category とする．

定義. lax 2-functor $\mathbf{1} \longrightarrow \mathcal{C}$ をモナドという．また $\mathcal{C}^{\mathrm{co}}$ でのモナド, 即ち oplax 2-functor $\mathbf{1} \longrightarrow \mathcal{C}$ をコモナドという．

$F\colon \mathbf{1} \longrightarrow \mathcal{C}$ をモナドとする．F は $\mathcal{C}$ の 2-morphism $\psi\colon \mathrm{id}_{F*} \Longrightarrow F(\mathrm{id}_*)$ と，次の自然変換 φ を与えられているのであった．

$$\begin{tikzcd} & \mathbf{1}(*,*) \times \mathbf{1}(*,*) \arrow[dl, "F\times F"'] \arrow[dr, "C"] & \\ \mathcal{C}(F*,F*) \times \mathcal{C}(F*,F*) \arrow[dr, "C"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\varphi}"] & & \mathbf{1}(*,*) \arrow[dl, "F"] \\ & \mathcal{C}(F*,F*) & \end{tikzcd}$$

故にモナド $F\colon \mathbf{1} \longrightarrow \mathcal{C}$ は対象 $a := F*$, 1-morphism $t := F(\mathrm{id}_*)\colon a \longrightarrow a$, 2-morphism $\mu := \varphi_{\mathrm{id}_*,\mathrm{id}_*}\colon t \circ t \Longrightarrow t$, $\eta\colon \mathrm{id}_a \Longrightarrow t$ の 4 つを与えると確定する．

$$\begin{tikzcd} & a \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mu"] & \\ a \arrow[ur, "t"] \arrow[rr, "t"'] & {} & a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, bend left, "\mathrm{id}_a"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] & a \end{tikzcd}$$

lax 2-functor の定義から，次の等号が成り立つ．

$$
\begin{tikzcd}[column sep=small]
 & a \arrow[r, "t"] \arrow[dr, phantom, "\Downarrow\mu"'] & a \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mu"] & \\
a \arrow[ur, "t"] \arrow[urr, "t"] \arrow[rrr, "t"'] & & & a
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & a \arrow[r, "t"] \arrow[d, phantom, "\Downarrow\mu"] & a \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mu"] & \\
a \arrow[ur, "t"] \arrow[rrr, "t"'] & & & a
\arrow[from=1-2, to=2-4, "t"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & a \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mu"] & \\
a \arrow[ur, bend left, "\mathrm{id}_a"] \arrow[ur, "t"'] \arrow[ur, phantom, bend left=20, "\Downarrow\eta"] \arrow[rr, "t"'] & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mathrm{id}_t"] & \\
a \arrow[ur, "\mathrm{id}_a"] \arrow[rr, "t"'] & & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & a \arrow[dr, "t"'] \arrow[dr, bend left, "\mathrm{id}_a"] \arrow[dr, phantom, bend left=20, "\Downarrow\eta"] \arrow[d, phantom, "\mu\Downarrow"] & \\
a \arrow[ur, "t"] \arrow[rr, "t"'] & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dr, "\mathrm{id}_a"] \arrow[d, phantom, "\Downarrow\mathrm{id}_t"] & \\
a \arrow[ur, "t"] \arrow[rr, "t"'] & & a
\end{tikzcd}
$$

逆に，a, t, μ, η が上記の条件を満たすとすれば，lax 2-functor $F\colon \mathbf{1} \longrightarrow \mathcal{C}$ を定義することができる．

こうして lax 2-functor $F\colon \mathbf{1} \longrightarrow \mathcal{C}$ と四つ組 $\langle a, t, \mu, \eta\rangle$ を同一視することができる．そこで，この条件を満たす $\langle a, t, \mu, \eta\rangle$ のこともモナドと呼ぶ．単に t のことをモナドということもある．

図式から容易に分かるように，$\mathcal{C}^{\mathrm{op}}$ でのモナドは $\mathcal{C}$ でのモナドでもある．

例. $a \in \mathcal{C}$ とするとき，$\langle a, \mathrm{id}_a, \mathrm{id}_{\mathrm{id}_a}, \mathrm{id}_{\mathrm{id}_a}\rangle$ はモナドである． □

例. $f\colon a \longrightarrow b$ として，左 Kan 拡張 $\langle f^\dagger f, \eta\rangle$ が存在するとする．このとき $f^\dagger f$ はコモナドになる．その為には，左 Kan 拡張の普遍性により次の μ と ε を取る．

$$
\begin{tikzcd}
b \arrow[r, "f^\dagger f"] \arrow[dr, "f^\dagger f"] & b \arrow[d, "f^\dagger f"] \arrow[d, phantom, shift right=4, "\mu\Uparrow"] \\
a \arrow[u, "f"] \arrow[r, "f"'] \arrow[ur, phantom, "\eta\Uparrow"'] & b
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[r, "f^\dagger f"] \arrow[d, phantom, shift left=4, "\eta\Uparrow"] & b \arrow[d, "f^\dagger f"] \arrow[d, phantom, shift right=4, "\eta\Uparrow"] \\
a \arrow[u, "f"] \arrow[r, "f"'] \arrow[ur, "f"] & b
\end{tikzcd}
$$

$$
\begin{tikzcd}
b \arrow[dr, "f^\dagger f"'] \arrow[dr, bend left, "\mathrm{id}_b"] \arrow[dr, phantom, bend left=20, "\Uparrow\varepsilon"] & \\
a \arrow[u, "f"] \arrow[r, "f"'] \arrow[ur, phantom, "\eta\Uparrow"'] & b
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[dr, bend left, "\mathrm{id}_b"] & \\
a \arrow[u, "f"] \arrow[r, "f"'] \arrow[ur, phantom, "\mathrm{id}_f\Uparrow"] & b
\end{tikzcd}
$$

このとき $\langle b, f^\dagger f, \mu, \varepsilon\rangle$ がコモナドとなることが分かる．この形のコモナドを density コモナドという．同様にして右 Kan 拡張 $f^\ddagger f$ はモナドとなり，これを codensity モナドという． $\square$

命題 1. $f \dashv u\colon a \longrightarrow b$ を随伴とし，unit を η，counit を ε とすれば，$\langle a, u\circ f, u\varepsilon_f, \eta\rangle$ はモナドである．

$$
\begin{tikzcd}
a \arrow[dr, "f"'] & & a \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & a \\
& b \arrow[ur, "u"] \arrow[rr, "\mathrm{id}_b"'] & {} & b \arrow[ur, "u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[rr, "\mathrm{id}_a"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & a \\
& b \arrow[ur, "u"'] &
\end{tikzcd}
$$

証明. まず結合律については

$$
\begin{tikzcd}
a \arrow[r, "u\circ f"] \arrow[dr, phantom, "\Downarrow u\varepsilon_f"] & a \arrow[d, "u\circ f"] \\
a \arrow[u, "u\circ f"] \arrow[ur, "u\circ f"] \arrow[r, "u\circ f"'] \arrow[ur, phantom, "\Downarrow u\varepsilon_f"] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[dr, "f"'] & & a \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & a \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & a \\
& b \arrow[ur, "u"] \arrow[rr, "\mathrm{id}_b"'] & {} & b \arrow[ur, "u"] \arrow[rr, "\mathrm{id}_b"'] & {} & b \arrow[ur, "u"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a \arrow[r, "u\circ f"] \arrow[dr, "u\circ f"] \arrow[d, phantom, "\Downarrow u\varepsilon_f"] & a \arrow[d, "u\circ f"] \arrow[dl, phantom, "\Leftarrow u\varepsilon_f"] \\
a \arrow[u, "u\circ f"] \arrow[r, "u\circ f"'] & a
\end{tikzcd}
$$

となり成り立つ．単位元についても

$$
\begin{tikzcd}
& a \arrow[dr, "u\circ f"] \arrow[d, phantom, "\Downarrow u\varepsilon_f"] & \\
a \arrow[ur, bend left, "\mathrm{id}_a"] \arrow[ur, "u\circ f"'] \arrow[ur, phantom, "\Downarrow\eta"] \arrow[rr, "u\circ f"'] & {} & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "\mathrm{id}_a"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & a \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & a \\
& b \arrow[ur, "u"] \arrow[rr, "\mathrm{id}_b"'] & {} & b \arrow[ur, "u"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
& a \arrow[dr, "u\circ f"] \arrow[d, phantom, "\Downarrow\mathrm{id}"] & \\
a \arrow[ur, "\mathrm{id}_a"] \arrow[rr, "u\circ f"'] & {} & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
& a \arrow[dr, "u\circ f"'] \arrow[dr, bend left, "\mathrm{id}_a"] \arrow[dr, phantom, "\Leftarrow\eta"] \arrow[d, phantom, "\Downarrow u\varepsilon_f"] & \\
a \arrow[ur, "u\circ f"] \arrow[rr, "u\circ f"'] & {} & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[dr, "f"'] & & a \arrow[rr, "\mathrm{id}_a"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & {} \arrow[d, phantom, "\Downarrow\eta"] & a \\
& b \arrow[ur, "u"] \arrow[rr, "\mathrm{id}_b"'] & {} & b \arrow[ur, "u"'] &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & a \arrow[dr, "\mathrm{id}_a"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & \\
a \arrow[ur, "u \circ f"] \arrow[rr, "u \circ f"'] & {} & a
\end{tikzcd}
$$

となり成り立つ. □

定義. $F, F' \colon \mathbf{1} \longrightarrow \mathcal{C}$ をモナドとする. F から F' へのモナド関手とは, pseudonatural transformation $F \Longrightarrow F'$ のことである.

モナド F を組 $\langle a, t, \mu, \eta \rangle$, モナド F' を組 $\langle a', t', \mu', \eta' \rangle$ と同一視したとき, $\langle a, t, \mu, \eta \rangle$ から $\langle a', t', \mu', \eta' \rangle$ へのモナド関手とは, 組 $\langle f, \varphi \rangle$ であって以下を満たすものであると言い換えることができる.

(1) $f \colon a \longrightarrow a'$ は 1-morphism で $\varphi \colon t' \circ f \Longrightarrow f \circ t$ は 2-morphism である.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow \varphi"] & a' \arrow[d, "t'"] \\
a \arrow[r, "f"'] & a'
\end{tikzcd}
$$

(2) $\varphi \circ \eta'_f = f\eta$ である.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow \varphi"] & a' \arrow[d, "t'"'] \arrow[d, bend left=60, "\mathrm{id}_{a'}"] \arrow[d, phantom, shift left=3, "\Leftarrow \eta'"] \\
a \arrow[r, "f"'] & a'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "t"'] \arrow[d, bend left=60, "\mathrm{id}_a"] \arrow[d, phantom, shift left=3, "\Leftarrow \eta"] & a' \\
a \arrow[r, "f"'] & a'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
{} \arrow[d, bend left=60, "\mathrm{id}_{a'}"] \\
{}
\end{tikzcd}
$$

(3) $\mu \circ \varphi_t \circ t'\varphi = \varphi \circ \mu'_f$ である.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "t"'] \arrow[dd, bend right=60, "t"'] \arrow[dr, phantom, "\swarrow \varphi"] & a' \arrow[d, "t'"] \\
a \arrow[r, "f"] \arrow[d, "t"'] \arrow[l, phantom, "\Leftarrow \mu"] \arrow[dr, phantom, "\swarrow \varphi"] & a' \arrow[d, "t'"] \\
a \arrow[r, "f"'] & a'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dd, bend right=60, "t"'] & & a' \arrow[d, "t'"] \arrow[dd, bend right=60, "t'"'] \\
 & \swarrow \varphi & a' \arrow[d, "t'"] \arrow[l, phantom, "\Leftarrow \mu'"] \\
a \arrow[rr, "f"'] & & a'
\end{tikzcd}
$$

$\langle a,t,\mu,\eta\rangle$ から $\langle a',t',\mu',\eta'\rangle$ へのモナド関手を $\langle f,\varphi\rangle\colon \langle a,t,\mu,\eta\rangle \longrightarrow \langle a',t',\mu',\eta'\rangle$，もしくは単に $f\colon t\longrightarrow t'$ で表す．

定義. $F,F'\colon \mathbf{1}\longrightarrow \mathcal{C}$ をモナド，$\sigma,\tau\colon F\Longrightarrow F'$ をモナド関手とする．σ から τ へのモナド関手変換とは modification $\sigma \Rrightarrow \tau$ のことである．

σ,τ を $\langle f,\varphi\rangle$，$\langle g,\psi\rangle$ と同一視すれば，$\langle f,\varphi\rangle$ から $\langle g,\psi\rangle$ へのモナド関手変換とは，2-mophism $\sigma\colon f\Longrightarrow g$ であって次の等式を満たすものである．

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow\varphi"] & a' \arrow[d, "t'"] \\ a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & a' \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow\psi"] & a' \arrow[d, "t'"] \\ a \arrow[r, bend right, "g"'] & a' \end{tikzcd}$$

$\mathrm{Monad}(\mathcal{C}) := \mathrm{Fun}_{\mathrm{lax}}(\mathbf{1},\mathcal{C})$ とすれば，$\mathrm{Monad}(\mathcal{C})$ は対象をモナド，1-morphism をモナド関手，2-morphism をモナド関手変換とする strict 2-category である．

$a,b\in\mathcal{C}$ を対象，$f,g\colon a\longrightarrow b$ を 1-morphism，$\varphi\colon f\Longrightarrow g$ を 2-morphism とする．$\mathrm{Inc}_{\mathcal{C}}(a) := \langle a,\mathrm{id}_a,\mathrm{id}_{\mathrm{id}_a},\mathrm{id}_{\mathrm{id}_a}\rangle$ はモナドであり $\mathrm{Inc}_{\mathcal{C}}(f) := \langle f,\mathrm{id}_f\rangle\colon \mathrm{Inc}_{\mathcal{C}}(a)\longrightarrow \mathrm{Inc}_{\mathcal{C}}(b)$ はモナド関手，$\mathrm{Inc}_{\mathcal{C}}(\varphi) := \varphi\colon f\Longrightarrow g$ はモナド関手変換である．こうして strict 2-functor $\mathrm{Inc}_{\mathcal{C}}\colon \mathcal{C}\longrightarrow \mathrm{Monad}(\mathcal{C})$ が得られる．

逆にモナド $\langle a,t,\mu,\eta\rangle$, モナド関手 $\langle f,\varphi\rangle$, モナド関手変換 σ に対して $U_{\mathcal{C}}(\langle a,t,\mu,\eta\rangle) := a$，$U_{\mathcal{C}}(\langle f,\varphi\rangle) := f$，$U_{\mathcal{C}}(\sigma) := \sigma$ と定めれば $U_{\mathcal{C}}\colon \mathrm{Monad}(\mathcal{C})\longrightarrow \mathcal{C}$ は strict 2-functor である．

命題 2. $\mathbf{Cat}$-随伴 $U_{\mathcal{C}} \dashv \mathrm{Inc}_{\mathcal{C}}\colon \mathrm{Monad}(\mathcal{C})\longrightarrow \mathcal{C}$ が成り立つ．

証明. 略 □

一般に，$\mathrm{Inc}_{\mathcal{C}}$ は右随伴を持つとは限らない．$\mathrm{Inc}_{\mathcal{C}}$ の右随伴が存在するとき，それを $\mathrm{Alg}_{\mathcal{C}}$ で表す．

定義. $\langle a,t,\mu,\eta\rangle$ をモナドとする．モナド関手 $\langle h,\xi\rangle\colon \mathrm{Inc}_{\mathcal{C}}(x)\longrightarrow t$ を左 t-加群という．即ち $\langle h,\xi\rangle$ は以下の条件を満たす．

- $h\colon x \longrightarrow a$ は 1-morphism で $\xi\colon t \circ h \Longrightarrow h$ は 2-morphism である．

$$
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & \\
 & a
\end{tikzcd}
$$

- 次の等式が成り立つ．

$$
\begin{tikzcd}
 & & a \arrow[dr, "t"] \arrow[d, phantom, "\xi\Downarrow"] & \\
x \arrow[urr, "h"] \arrow[rrr, "h"] \arrow[drr, "h"'] & & {} \arrow[d, phantom, "\xi\Downarrow"] & a \arrow[dl, "t"] \\
 & & a &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & a \arrow[dd, bend right, "t"'] \arrow[dr, "t"] & \\
x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} & a \arrow[dl, "t"] \arrow[l, phantom, "\overset{\mu}{\Longleftarrow}"] \\
 & a &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & a \arrow[dr, "t"'] \arrow[dr, bend left=60, "\mathrm{id}_a"] \arrow[d, phantom, "\xi\Downarrow"] \arrow[dr, phantom, bend left=30, "\eta\Swarrow"] & \\
x \arrow[ur, "h"] \arrow[rr, "h"'] & {} & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & a \arrow[dr, bend left=60, "\mathrm{id}_a"] \arrow[d, phantom, "\Downarrow\mathrm{id}_h"] & \\
x \arrow[ur, "h"] \arrow[rr, "h"'] & {} & a
\end{tikzcd}
$$

定義. モナド $\langle a, t, \mu, \eta\rangle$ の普遍左 t-加群とは左 t-加群 $\langle u\colon a^t \to a,\ \varphi\colon t \circ u \Rightarrow u\rangle$ であって，以下の条件を満たすものをいう．

- $\langle h\colon x \to a,\ \xi\colon t \circ h \Rightarrow h\rangle$ が左 t-加群ならば，1-morphism $f\colon x \longrightarrow a^t$ が一意に存在して次の等式が成り立つ．

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] \arrow[r, dashed, "f"] & a^t \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, phantom, "\varphi\Downarrow"] & {} \\
 & & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

- $\langle h, \xi\rangle$，$\langle h', \xi'\rangle$ を左 t-加群とする．上記の条件により $f, f'\colon x \longrightarrow a^t$ が取れる．

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] \arrow[r, dashed, "f"] & a^t \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, phantom, "\varphi\Downarrow"] & {} \\
 & & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[r, dashed, "f'"] \arrow[urr, bend left, "h'"] \arrow[drr, bend right, "h'"'] & a^t \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, phantom, "\varphi\Downarrow"] & {} \\
 & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h'"] \arrow[dr, bend right, "h'"'] \arrow[r, phantom, "\xi'\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

2-morphism $\sigma\colon h \Longrightarrow h'$ が次の等式を満たすとき

$$
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left=40, "h"] \arrow[ur, bend right=20, "h'"'] \arrow[dr, bend right, "h'"'] \arrow[ur, phantom, "\Downarrow\sigma"] \arrow[dr, phantom, "\xi'\Downarrow"] & {} \\
 & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend left=20, "h"] \arrow[dr, bend right=40, "h'"'] \arrow[ur, phantom, "\xi\Downarrow"] \arrow[dr, phantom, "\Downarrow\sigma"] & {} \\
 & a
\end{tikzcd}
$$

2-morphism $\tau\colon f \Longrightarrow f'$ が一意に存在して $u \circ \tau = \sigma$ となる．

$$
\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\tau"] & a^t \arrow[r, "u"] & b
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\sigma"] & b
\end{tikzcd}
$$

また a^t をモナド t の Eilenberg-Moore 対象という．

定理 3. $\mathrm{Alg}_{\mathcal{C}}$ が存在するならば，任意のモナド $\langle a, t, \mu, \eta\rangle$ に対して $\mathrm{Alg}_{\mathcal{C}}(t)$ はモナド t の Eilenberg-Moore 対象である．

証明. 随伴 $\mathrm{Inc}_{\mathcal{C}} \dashv \mathrm{Alg}_{\mathcal{C}}$ の counit が定めるモナド関手を $\langle u^t, \delta^t\rangle\colon \mathrm{Inc}_{\mathcal{C}}(\mathrm{Alg}_{\mathcal{C}}(t)) \longrightarrow t$ とする．即ち $\langle u^t, \delta^t\rangle$ は左 t-加群である．

$$
\begin{tikzcd}
\mathrm{Alg}_{\mathcal{C}}(t) \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow\delta^t"] & a \arrow[d, "t"] \\
\mathrm{Alg}_{\mathcal{C}}(t) \arrow[r, "u^t"'] & a
\end{tikzcd}
$$

$\langle u^t, \delta^t\rangle$ が普遍左 t-加群であることを示すため，$\langle h, \xi\rangle$ を左 t-加群とする．

$$
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

即ち $\langle h, \xi \rangle$ はモナド関手 $\mathrm{Inc}_{\mathcal{C}}(x) \longrightarrow t$ である．

$$
\begin{tikzcd}
x \arrow[r, "h"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow\xi"] & a \arrow[d, "t"] \\
x \arrow[r, "h"'] & a
\end{tikzcd}
$$

随伴 $\mathrm{Inc}_{\mathcal{C}} \dashv \mathrm{Alg}_{\mathcal{C}}$ で $\langle h, \xi \rangle \colon \mathrm{Inc}_{\mathcal{C}}(x) \longrightarrow t$ に対応する $f \colon x \longrightarrow \mathrm{Alg}_{\mathcal{C}}(t)$ を取る．$\langle u^t, \delta^t \rangle$ の取り方から，次の等式が成り立つ．

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow\mathrm{id}_f"] & \mathrm{Alg}_{\mathcal{C}}(t) \arrow[r, "u^t"] \arrow[d, "\mathrm{id}_a"] \arrow[dr, phantom, "\swarrow\delta^t"] & a \arrow[d, "t"] \\
x \arrow[r, "f"'] & \mathrm{Alg}_{\mathcal{C}}(t) \arrow[r, "u^t"'] & a
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[rr, "h"] \arrow[d, "\mathrm{id}_b"'] \arrow[drr, phantom, "\swarrow\xi"] & & a \arrow[d, "t"] \\
x \arrow[rr, "h"'] & & a
\end{tikzcd}
$$

これは書きかえれば次の図式となる．随伴 $\mathrm{Inc}_{\mathcal{C}} \dashv \mathrm{Alg}_{\mathcal{C}}$ の性質から f が一意であることも分かる．

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] \arrow[r, dashed, "f"] & \mathrm{Alg}_{\mathcal{C}}(t) \arrow[ur, "u^t"'] \arrow[dr, "u^t"] \arrow[r, phantom, "\delta^t\Downarrow"] & {} \\
 & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

次に $\langle h, \xi \rangle$，$\langle h', \xi' \rangle$ を左 t-加群として 2-morphism $\sigma \colon h \Longrightarrow h'$ が次の等式を満たすとする．

$$
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[ur, bend right, "h'"'] \arrow[dr, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\sigma"] & {} \\
 & a \arrow[u, phantom, "\xi'\Downarrow"]
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend left, "h"] \arrow[dr, bend right, "h'"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a \arrow[l, phantom, "\swarrow\sigma"]
\end{tikzcd}
$$

このとき σ はモナド関手変換 $\langle h, \xi \rangle \Longrightarrow \langle h', \xi' \rangle$ を与える．よって随伴 $\mathrm{Inc}_{\mathcal{C}} \dashv \mathrm{Alg}_{\mathcal{C}}$ による圏同型

$$\mathrm{Monad}(\mathcal{C})(\mathrm{Inc}_{\mathcal{C}}(x), t) \cong \mathcal{C}(x, \mathrm{Alg}_{\mathcal{C}}(t))$$

で $\sigma \colon \langle h, \xi \rangle \Longrightarrow \langle h', \xi' \rangle \colon \mathrm{Inc}_{\mathcal{C}}(x) \longrightarrow t$ に対応する $\tau \colon f \Longrightarrow f' \colon x \longrightarrow a^t$ が取れる．このとき counit の性質から $u^t \circ \tau = \sigma$ であり，このような τ は一意である．

以上により $\langle u^t, \delta^t \rangle$ は普遍左 t-加群である． □

逆に

定理 4. 任意のモナド t に対して Eilenberg-Moore 対象が存在するならば $\mathrm{Alg}_{\mathcal{C}}$ が存在する．

証明. まず以下のように定義する．

- モナド $\langle a, t, \mu, \eta \rangle$ に対して $\mathrm{Alg}_{\mathcal{C}}(t) := a^t$ と定める．
- $\langle a, t, \mu, \eta \rangle$，$\langle a', t', \mu', \eta' \rangle$ をモナド，$\langle f, \varphi \rangle \colon t \longrightarrow t'$ をモナド関手とする．$a^{t'}$ の普遍性により $\mathrm{Alg}_{\mathcal{C}}(\langle f, \varphi \rangle) \colon a^t \longrightarrow a^{t'}$ を定める．

$$
\begin{tikzcd}
 & a \arrow[r, "f"] & a' \arrow[dd, "t'"] \\
\mathrm{Alg}_{\mathcal{C}}(t) \arrow[ur, bend left, "u^t"] \arrow[dr, bend right, "u^t"'] \arrow[r, dashed, "\mathrm{Alg}_{\mathcal{C}}(f)"] & \mathrm{Alg}_{\mathcal{C}}(t') \arrow[u, equal] \arrow[d, equal] \arrow[ur, "u^{t'}"] \arrow[dr, "u^{t'}"'] \arrow[r, phantom, "\delta^{t'}\Downarrow"] & \\
 & a \arrow[r, "f"'] & a'
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[dd, "t"] \arrow[ddr, phantom, "\Swarrow\varphi"] & a' \arrow[dd, "t'"] \\
\mathrm{Alg}_{\mathcal{C}}(t) \arrow[ur, bend left, "u^t"] \arrow[dr, bend right, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & & \\
 & a \arrow[r, "f"'] & a'
\end{tikzcd}
$$

- $\langle a, t, \mu, \eta \rangle$，$\langle a', t', \mu', \eta' \rangle$ をモナド，$\langle f, \varphi \rangle \colon t \longrightarrow t'$ をモナド関手，$\sigma \colon \langle f, \varphi \rangle \Longrightarrow \langle f', \varphi' \rangle$ をモナド関手変換とする．このとき σ は次の等式を満たす．

$$
\begin{tikzcd}
 & a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\sigma"] \arrow[d, "t"] \arrow[dr, phantom, "\Swarrow\varphi'"] & a' \arrow[d, "t'"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\xi\Downarrow"] & a \arrow[r, bend right, "f'"'] & a'
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, bend left, "f"] \arrow[d, "t"] \arrow[dr, phantom, "\Swarrow\varphi"] & a' \arrow[d, "t'"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\xi\Downarrow"] & a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\sigma"] & a'
\end{tikzcd}
$$

  よって $\mathrm{Alg}_{\mathcal{C}}(\sigma) \colon \mathrm{Alg}_{\mathcal{C}}(\langle f, \varphi \rangle) \Longrightarrow \mathrm{Alg}_{\mathcal{C}}(\langle f', \varphi' \rangle)$ が一意に存在して $u^{t'} \circ \mathrm{Alg}_{\mathcal{C}}(\sigma) = \sigma \circ u^t$ となる．

これは strict 2-functor $\mathrm{Alg}_{\mathcal{C}} \colon \mathrm{Monad}(\mathcal{C}) \longrightarrow \mathcal{C}$ を与えることが分かる．$\mathrm{Inc}_{\mathcal{C}} \dashv \mathrm{Alg}_{\mathcal{C}}$ を示せばよい．その為には x, t について自然な圏同型

$$\Phi_{xt} \colon \mathrm{Monad}(\mathcal{C})(\mathrm{Inc}_{\mathcal{C}}(x), t) \cong \mathcal{C}(x, a^t)$$

を構成すればよい．Φ_{xt} を a^t の普遍性により自然に定めれば，Φ_{xt} は圏同型を与える．よってこの Φ_{xt} が自然であればよい．以下, 簡単のため $\mathrm{M}(x, t) := \mathrm{Monad}(\mathcal{C})(\mathrm{Inc}_{\mathcal{C}}(x), t)$ と書く．

まず x について自然であることを示す．その為には 2-morphism $\alpha\colon f \Longrightarrow g\colon x \longrightarrow x'$ に対して等式

$$\begin{tikzcd}[column sep=small]
\mathrm{M}(x',t) \arrow[r, "\Phi_{x't}"] & \mathcal{C}(x',a^t) \arrow[r, bend left, "-\circ f"] \arrow[r, bend right, "-\circ g"'] \arrow[r, phantom, "\Downarrow -\circ\alpha"] & \mathcal{C}(x,a^t) & = & \mathrm{M}(x',t) \arrow[r, bend left, "-\circ\langle f,\mathrm{id}_f\rangle"] \arrow[r, bend right, "-\circ\langle g,\mathrm{id}_g\rangle"'] \arrow[r, phantom, "\Downarrow -\circ\alpha"] & \mathrm{M}(x,t) \arrow[r, "\Phi_{xt}"] & \mathcal{C}(x,a^t)
\end{tikzcd}$$

を示せばよい．まず $\Phi_{x't}(-)\circ f = \Phi_{xt}(-\circ\langle f,\mathrm{id}_f\rangle)$ を示す．

$\because$) $\langle h,\xi\rangle\colon \mathrm{Inc}_{\mathcal{C}}(x') \longrightarrow t$ をモナド関手，すなわち左 t-加群とする．普遍性から $x' \longrightarrow a^t$ が取れて，これが $\Phi_{x't}(h)$ である．

$$\begin{tikzcd}
& & a \arrow[dd, "t"] \\
x' \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] \arrow[r, dashed, "\Phi_{x't}(h)"] & a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & {} \\
& & a
\end{tikzcd}
=
\begin{tikzcd}
& & a \arrow[dd, "t"] \\
x' \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & {} \arrow[r, phantom, "\xi\Downarrow"] & {} \\
& & a
\end{tikzcd}$$

一方，左 t-加群 $h\circ f$ に対応するのが $\Phi_{xt}(h\circ f)$ である．

$$\begin{tikzcd}
& x' \arrow[r, "h"] & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f"'] \arrow[r, dashed, "\Phi_{xt}(h\circ f)"] & a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & {} \\
& x' \arrow[r, "h"'] & a
\end{tikzcd}
=
\begin{tikzcd}
& x' \arrow[r, "h"] \arrow[dd, "\mathrm{id}_{x'}"'] \arrow[ddr, phantom, "\swarrow\xi"] & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f"'] \arrow[r, phantom, "\mathrm{id}_f\Downarrow"] & {} & {} \\
& x' \arrow[r, "h"'] & a
\end{tikzcd}$$

よって普遍性から $\Phi_{x't}(h)\circ f = \Phi_{xt}(h\circ f)$ となることが分かる．次に $\sigma\colon \langle h,\xi\rangle \Longrightarrow \langle h',\xi'\rangle\colon \mathrm{Inc}_{\mathcal{C}}(x') \longrightarrow t$ をモナド関手変換とする．$\Phi_{x't}(\sigma)$ は次の等式を満たす．

$$\begin{tikzcd}
x' \arrow[r, bend left, "\Phi_{x't}(h)"] \arrow[r, bend right, "\Phi_{x't}(h')"'] \arrow[r, phantom, "\Downarrow\Phi_{x't}(\sigma)"] & a^t \arrow[r, "u^t"] & a
\end{tikzcd}
=
\begin{tikzcd}
x' \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\sigma"] & a
\end{tikzcd}$$

一方 $\Phi_{xt}(\sigma\circ f)$ は次の等式を満たす．

$$
\begin{tikzcd}
x' \arrow[r, bend left, "\Phi_{xt}(h\circ f)"] \arrow[r, bend right, "\Phi_{xt}(h'\circ f)"'] \arrow[r, phantom, "\Downarrow\Phi_{xt}(\sigma\circ f)"] & a^t \arrow[r, "u^t"] & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "f"] & x' \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\sigma"] & a
\end{tikzcd}
$$

よって普遍性から $\Phi_{x't}(\sigma)\circ f=\Phi_{xt}(\sigma\circ f)$ である．

以上により $\Phi_{x't}(-)\circ f=\Phi_{xt}(-\circ\langle f,\mathrm{id}_f\rangle)$ である．

次に自然変換の等式 $\Phi_{x't}(-)\circ\alpha=\Phi_{xt}(-\circ\alpha)$ を示す．その為には左 t-加群 $\langle h,\xi\rangle$ に対して $\Phi_{x't}(h)\circ\alpha=\Phi_{xt}(h\circ\alpha)$ を示せばよいが，これも普遍性から簡単にわかる．

次に t について自然であることを示す．その為にはモナド関手変換 $\sigma\colon\langle f,\varphi\rangle\Longrightarrow\langle g,\psi\rangle\colon t\longrightarrow t'$ に対して等式

$$
\begin{tikzcd}
\mathrm{M}(x,t) \arrow[r, "\Phi_{xt}"] & \mathcal{C}(x,a^t) \arrow[r, bend left, "\mathrm{Alg}(f)\circ-"] \arrow[r, bend right, "\mathrm{Alg}(g)\circ-"'] \arrow[r, phantom, "\Downarrow\mathrm{Alg}(\sigma)\circ-"] & \mathcal{C}(x,a^{t'})
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{M}(x,t) \arrow[r, bend left, "\langle f,\varphi\rangle\circ-"] \arrow[r, bend right, "\langle g,\psi\rangle\circ-"'] \arrow[r, phantom, "\Downarrow\sigma\circ-"] & \mathrm{M}(x,t') \arrow[r, "\Phi_{xt'}"] & \mathcal{C}(x,a^{t'})
\end{tikzcd}
$$

を示せばよい．まず $\mathrm{Alg}_{\mathcal{C}}(f)\circ\Phi_{xt}(-)=\Phi_{xt'}(\langle f,\varphi\rangle\circ-)$ を示す．

$\because$) $\langle h,\xi\rangle\colon\mathrm{Inc}_{\mathcal{C}}(x)\longrightarrow t$ をモナド関手，すなわち左 t-加群とする．$\Phi_{xt}(h),\mathrm{Alg}_{\mathcal{C}}(f)$ は次の等式で与えられる．

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] \arrow[r, dashed, "\Phi_{xt}(h)"] & a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & {} \\
 & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & a \arrow[r, "f"] & a' \arrow[dd, "t'"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, "\mathrm{Alg}_{\mathcal{C}}(f)"] & a^{t'} \arrow[ur, "u^{t'}"] \arrow[dr, "u^{t'}"'] \arrow[r, phantom, "\delta^{t'}\Downarrow"] & {} \\
 & a \arrow[r, "f"'] & a'
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[dd, "t"] \arrow[ddr, phantom, "\swArrow\varphi"] & a' \arrow[dd, "t'"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & {} & \\
 & a \arrow[r, "f"'] & a'
\end{tikzcd}
$$

よって $\mathrm{Alg}_{\mathcal{C}}(f) \circ \Phi_{xt}(h)$ は

$$
\begin{tikzcd}
 & & a \arrow[rr, "f"] & & a' \arrow[dd, "t'"] \\
x \arrow[urr, bend left, "h"] \arrow[r, "\Phi_{xt}(h)"] \arrow[drr, bend right, "h"'] & a^t \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[rr, "\mathrm{Alg}_{\mathcal{C}}(f)"] & & a^{t'} \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u^{t'}"] \arrow[dr, "u^{t'}"'] \arrow[r, phantom, "\delta^{t'}\Downarrow"] & \\
 & & a \arrow[rr, "f"'] & & a'
\end{tikzcd}
=
\begin{tikzcd}
 & & a \arrow[r, "f"] \arrow[dd, "t"] \arrow[ddr, phantom, "\swarrow\varphi"] & a' \arrow[dd, "t'"] \\
x \arrow[urr, bend left, "h"] \arrow[r, "\Phi_{xt}(h)"] \arrow[drr, bend right, "h"'] & a^t \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & & \\
 & & a \arrow[r, "f"'] & a'
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
 & a \arrow[r, "f"] \arrow[dd, "t"] \arrow[ddr, phantom, "\swarrow\varphi"] & a' \arrow[dd, "t'"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & & \\
 & a \arrow[r, "f"'] & a'
\end{tikzcd}
$$

より $\mathrm{Alg}_{\mathcal{C}}(f) \circ \Phi_{xt}(h) = \mathrm{Alg}_{\mathcal{C}}(f \circ h)$ となることが分かる．次に $\sigma\colon \langle h, \xi \rangle \Longrightarrow \langle h', \xi' \rangle\colon \mathrm{Inc}_{\mathcal{C}}(x) \longrightarrow t$ をモナド関手変換とする．$\Phi_{xt}(\sigma)$，$\Phi_{xt'}(f \circ \sigma)$ は次の等式で与えられる．

$$
\begin{tikzcd}
x \arrow[r, bend left, "\Phi_{xt}(h)"] \arrow[r, bend right, "\Phi_{xt}(h')"'] \arrow[r, phantom, "\Downarrow \Phi_{xt}(\sigma)"] & a^t \arrow[r, "u^t"] & a
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow \sigma"] & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \arrow[r, bend left, "\Phi_{xt'}(f \circ h)"] \arrow[r, bend right, "\Phi_{xt'}(f \circ h')"'] \arrow[r, phantom, "\Downarrow \Phi_{xt'}(f \circ \sigma)"] & a^{t'} \arrow[r, "u^t"] & a'
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow \sigma"] & a \arrow[r, "f"] & a'
\end{tikzcd}
$$

よって $\mathrm{Alg}_{\mathcal{C}}(f) \circ \Phi_{xt}(\sigma) = \Phi_{xt'}(f \circ \sigma)$ である．

以上により $\mathrm{Alg}_{\mathcal{C}}(f) \circ \Phi_{xt}(-) = \Phi_{xt'}(\langle f, \varphi \rangle \circ -)$ である．

後は自然変換について示せばよい．つまり左 t-加群 $\langle h, \xi \rangle$ に対して $\mathrm{Alg}_{\mathcal{C}}(\sigma) \circ \Phi_{xt}(h) = \Phi_{xt'}(\sigma \circ h)$ を示せばよいがこれも普遍性から分かる． □

よって，以下 $a^t = \mathrm{Alg}_{\mathcal{C}}(t)$ と書くことにする．

定理 5． $\mathrm{Alg}_{\mathcal{C}}$ が存在するならば，任意のモナド $\langle a, t, \mu, \eta \rangle$ は随伴から命題 1 の方法で得られる．

証明. $\langle t,\mu\rangle$ は左 t-加群を定める.

$$
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
a \arrow[ur, "t"] \arrow[dr, "t"'] \arrow[r, phantom, "\mu\Downarrow"] & \\
 & a
\end{tikzcd}
$$

よって $f^t\colon a \longrightarrow a^t$ が一意に存在して次の等式が成り立つ.

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
a \arrow[r, dashed, "f^t"] \arrow[urr, bend left, "t"] \arrow[drr, bend right, "t"'] & a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & \\
 & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
a \arrow[ur, bend left, "t"] \arrow[dr, bend right, "t"'] \arrow[r, phantom, "\mu\Downarrow"] & \\
 & a
\end{tikzcd}
$$

特に $t = u^t \circ f^t$ である．故に，$f^t \dashv u^t$ を示して，この随伴から命題 1 の方法で得たモナドが $\langle a,t,\mu,\eta\rangle$ と一致することを示せばよい.

$\delta^t\colon t\circ u^t \Longrightarrow u^t$ はモナド関手変換 $\langle t,\mu\rangle \circ \langle u^t, \mathrm{id}_{u^t}\rangle \Longrightarrow \langle u^t,\delta^t\rangle$ を定める.

$\because$)

$$
\begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}_{a^t}"'] \arrow[dr, phantom, "\Swarrow\mathrm{id}_{u^t}"] & a \arrow[r, "t"] \arrow[d, "\mathrm{id}_a"'] \arrow[dr, phantom, "\Swarrow\mu"] & a \arrow[d, "t"] \\
a^t \arrow[r, "u^t"'] \arrow[rr, bend right, "u^t"', ""{name=U}] & a \arrow[r, "t"'] \arrow[phantom, d, "\Downarrow\delta^t"] & a \\
 & {} &
\end{tikzcd}
=
\begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[dd, "\mathrm{id}_{a^t}"'] \arrow[ddr, phantom, "\Swarrow\delta^t"] & a \arrow[d, "t"] \arrow[dd, bend right=60, "t"'] \\
 & a \arrow[d, "t"] \arrow[l, phantom, "\overset{\mu}{\Leftarrow}"] \\
a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow\delta^t"] \arrow[dd, bend right=60, "\mathrm{id}"'] & a \arrow[d, "t"] \\
a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow\delta^t"] \arrow[l, phantom, "\overset{\mathrm{id}}{\Leftarrow}"] & a \arrow[d, "t"] \\
a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\delta^t"] & \\
a^t \arrow[ur, "u^t"] \arrow[rr, "u^t"] \arrow[d, "\mathrm{id}"] \arrow[drr, phantom, "\Swarrow\delta^t"] & {} & a \arrow[d, "t"] \\
a^t \arrow[rr, "u^t"'] & & a
\end{tikzcd}
$$

よって，随伴による圏同型

$$
\mathrm{Monad}(\mathcal{C})(\mathrm{Inc}_{\mathcal{C}}(a^t), t) \cong \mathcal{C}(a^t, a^t)
$$

で $\delta^t\colon t\circ u^t \Longrightarrow u^t\colon \mathrm{Inc}_{\mathcal{C}}(a^t)\longrightarrow t$ に対応する $\varepsilon^t\colon k\Longrightarrow \mathrm{id}\colon a^t\longrightarrow a^t$ が取れる．counit の性質から $\langle u^t,\delta^t\rangle\circ\langle k,\mathrm{id}\rangle = \langle t,\mu\rangle\circ\langle u^t,\mathrm{id}_{u^t}\rangle$，$u^t\circ\varepsilon=\delta^t$ である．

$$
\begin{tikzcd}
a^t \arrow[r, "k"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow\mathrm{id}_k"] & a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}_a"] \arrow[dr, phantom, "\Swarrow\delta^t"] & a \arrow[d, "t"] \\
a^t \arrow[r, "k"'] & a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
=
\begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow\mathrm{id}_{u^t}"] & a \arrow[r, "t"] \arrow[d, "\mathrm{id}_a"] \arrow[dr, phantom, "\Swarrow\mu"] & a \arrow[d, "t"] \\
a^t \arrow[r, "u^t"'] & a \arrow[r, "t"'] & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
a^t \arrow[r, bend left, "k"] \arrow[r, bend right, "\mathrm{id}_{a^t}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & a^t \arrow[r, "u^t"] & a
\end{tikzcd}
=
\begin{tikzcd}
a^t \arrow[r, bend left, "u^t"] \arrow[rr, bend right, "u^t"'] & a^t \arrow[r, bend left, "t"] \arrow[d, phantom, "\Downarrow\delta^t"] & a \\
 & &
\end{tikzcd}
$$

$t=u^t\circ f^t$ だから $u^t\circ k = u^t\circ f^t\circ u^t$ であり，よって u^t の普遍性から $k=f^t\circ u^t$ が分かる．また $u^t\varepsilon_{f^t}=\delta^t_{f^t}=\mu$ である．

後は η,ε^t が随伴 $f^t\dashv u^t\colon a\longrightarrow a^t$ を与えることを示せばよい．まず

$$
\begin{tikzcd}
a \arrow[rr, "\mathrm{id}"] \arrow[dr, "f^t"'] & \arrow[d, phantom, "\Downarrow\eta"] & a \arrow[rr, "\mathrm{id}"] \arrow[dr, "f^t"] \arrow[d, phantom, "\Downarrow\varepsilon^t"] & \arrow[d, phantom, "\Vert"] & a \\
 & a^t \arrow[ur, "u^t"] \arrow[rr, "\mathrm{id}"'] & & a^t \arrow[ur, "u^t"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a \arrow[rr, "\mathrm{id}"] \arrow[dr, "f^t"'] & \arrow[d, phantom, "\Downarrow\eta"] & a \arrow[rr, "t"] \arrow[drr, phantom, "\Downarrow\delta^t"] & & a \\
 & a^t \arrow[ur, "u^t"] \arrow[rr, "\mathrm{id}"'] & & a^t \arrow[ur, "u^t"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a \arrow[rr, bend left=50, "\mathrm{id}"] \arrow[r, "f^t"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow\mathrm{id}_{f^t}"] & a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"] \arrow[u, phantom, "\Downarrow\eta"] \arrow[dr, phantom, "\Swarrow\delta^t"] & a \arrow[d, "t"] \\
a \arrow[r, "f^t"'] & a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, phantom, shift left=4ex, "\Downarrow\eta"] \arrow[r, "t"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow\mu"] & a \arrow[d, "t"] \\
a \arrow[r, "t"'] & a
\end{tikzcd}
$$

$$
= \begin{tikzcd}
a \arrow[r, bend left, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_t"] & a \arrow[d, "t"] \\
a \arrow[r, "t"'] & a
\end{tikzcd}
$$

より $u^t(\varepsilon^t_{f^t} \circ f^t\eta) = \mathrm{id}_t$ であるから，$\varepsilon_{f^t} \circ f^t\eta = \mathrm{id}_{f^t}$ が分かる．また

$$
\begin{tikzcd}
 & a \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "f^t"] \arrow[r, phantom, "\Downarrow\eta"] \arrow[d, phantom, "\Downarrow\varepsilon^t"] & & a \\
a^t \arrow[ur, "u^t"] \arrow[rr, "\mathrm{id}"'] & & a^t \arrow[ur, "u^t"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[rr, bend left, "\mathrm{id}"] \arrow[rr, "t"] \arrow[r, phantom, "\Downarrow\eta"] \arrow[dr, phantom, "\Downarrow\delta^t"] & & a \\
a^t \arrow[ur, "u^t"] \arrow[rr, "\mathrm{id}"'] & & a^t \arrow[ur, "u^t"'] &
\end{tikzcd}
$$

$$
= \begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow\delta^t"] & a \arrow[d, "t"'] \arrow[d, bend left=60, "\mathrm{id}"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] \\
a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
= \begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[d, "\mathrm{id}"'] \arrow[d, bend left=60, "\mathrm{id}"] \arrow[d, phantom, "\overset{\mathrm{id}}{\Leftarrow}"] & a \\
a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
= \begin{tikzcd}
a \arrow[d, bend left=60, "\mathrm{id}"] \\
a
\end{tikzcd}
$$

より $u^t\varepsilon^t \circ \eta_{u^t} = \mathrm{id}_{u^t}$ である． □

定理 6. $\mathrm{Alg}_{\mathcal{C}}$ が存在するとする．随伴 $f \dashv u \colon a \longrightarrow b$ の unit を η，counit を ε とする．$f \dashv u$ から得られるモナドを $\langle a, t, \mu, \eta \rangle$ とする．定理 5 により，モナド t から随伴 $f^t \dashv u^t \colon a \longrightarrow a^t$ (unit は η，counit は ε^t) が得られる．このとき 1-morphism $h \colon b \longrightarrow a^t$ が一意に存在して $u^t \circ h = u$，$u^t\varepsilon_h = (u^t \circ h)\varepsilon$ となる．更に $f \circ h = f^t$ かつ $h\varepsilon = \varepsilon^t_h$ である．

$$
\begin{tikzcd}
 & a \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "f^t"] & \\
b \arrow[ur, bend right, "u"'] \arrow[ur, phantom, "\dashv"] \arrow[rr, dashed, "h"'] & & a^t \arrow[ul, bend right, "u^t"'] \arrow[ul, phantom, "\dashv"]
\end{tikzcd}
$$

証明. $\langle u, u\circ\varepsilon\rangle\colon \mathrm{Inc}_{\mathcal{C}}(b)\longrightarrow t$ はモナド関手，よって左 t-加群である．

$$
\begin{tikzcd}
b \arrow[r, "u"] \arrow[dd, "\mathrm{id}_b"'] \arrow[dr, "\mathrm{id}_b"] & a \arrow[d, "f"] \arrow[d, phantom, shift right=3ex, "\varepsilon\Downarrow"] \\
 & b \arrow[d, "u"] \arrow[dl, phantom, "\mathrm{id}_u\Downarrow"] \\
b \arrow[r, "u"'] & a
\end{tikzcd}
$$

故に $h\colon b\longrightarrow a^t$ が存在して次の等式が成り立つ．

$$
\begin{tikzcd}
b \arrow[r, "h"] \arrow[dd, "\mathrm{id}_b"'] \arrow[ddr, phantom, "\mathrm{id}_h\swarrow"] & a^t \arrow[r, "u^t"] \arrow[dd, "\mathrm{id}"'] \arrow[ddr, phantom, "\delta^t\swarrow"] & a \arrow[d, "f"] \\
 & & b \arrow[d, "u"] \\
b \arrow[r, "h"'] & a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[r, "u"] \arrow[dd, "\mathrm{id}_b"'] \arrow[dr, "\mathrm{id}_b"] & a \arrow[d, "f"] \arrow[d, phantom, shift right=3ex, "\varepsilon\Downarrow"] \\
 & b \arrow[d, "u"] \arrow[dl, phantom, "\mathrm{id}_u\Downarrow"] \\
b \arrow[r, "u"'] & a
\end{tikzcd}
$$

従って $u^t\circ h=u$，$\delta^t_h=u\varepsilon$ である．$\delta^t=u^t\varepsilon$ だったから $u^t\varepsilon_h=(u^t\circ h)\varepsilon$ となる．

$$
\begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[dd, "\mathrm{id}"'] \arrow[ddr, phantom, "\delta^t\swarrow"] & a \arrow[d, "f"] \\
 & b \arrow[d, "u"] \\
a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
=
\begin{tikzcd}
a^t \arrow[r, "u^t"] \arrow[dd, "\mathrm{id}"'] \arrow[dd, phantom, shift left=3ex, "\Downarrow\varepsilon^t"] & a \arrow[d, "f"] \arrow[ddl, "f^t"] \\
 & b \arrow[d, "u"] \arrow[d, phantom, shift right=3ex, "\parallel"] \\
a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
$$

このような h の一意性は u^t の普遍性から分かる．

t の定義から $\mu=u\varepsilon_f$ だったから

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dd, "\mathrm{id}_a"'] \arrow[ddr, phantom, "\mathrm{id}_f\swarrow"] & b \arrow[r, "h"] \arrow[dd, "\mathrm{id}_b"'] \arrow[ddr, phantom, "\mathrm{id}_h\swarrow"] & a^t \arrow[r, "u^t"] \arrow[dd, "\mathrm{id}_{a^t}"'] \arrow[ddr, phantom, "\delta^t\swarrow"] & a \arrow[d, "f"] \\
 & & & b \arrow[d, "u"] \\
a \arrow[r, "f"'] & b \arrow[r, "h"'] & a^t \arrow[r, "u^t"'] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dd, "\mathrm{id}_a"'] \arrow[ddr, phantom, "\mathrm{id}_f\swarrow"] & b \arrow[r, "u"] \arrow[dd, "\mathrm{id}_b"'] \arrow[dr, "\mathrm{id}_b"] & a \arrow[d, "f"] \arrow[d, phantom, shift right=3ex, "\varepsilon\Downarrow"] \\
 & & b \arrow[d, "u"] \arrow[dl, phantom, "\mathrm{id}_u\Downarrow"] \\
a \arrow[r, "f"'] & b \arrow[r, "u"'] & a
\end{tikzcd}
$$

$$
= \begin{tikzcd} a \arrow[r,"f"] \arrow[dd,"\mathrm{id}_a"'] & b \arrow[r,"u"] \arrow[dd, phantom, "\overset{\mu}{\swarrow}"] & a \arrow[d,"f"] \\ & & b \arrow[d,"u"] \\ a \arrow[r,"f"'] & b \arrow[r,"u"'] & a \end{tikzcd}
= \begin{tikzcd} & & a \arrow[dd,"t"] \\ a \arrow[r,"f^t"] \arrow[urr, bend left, "t"] \arrow[drr, bend right, "t"'] & a^t \arrow[ur,"u^t"] \arrow[dr,"u^t"'] \arrow[r, phantom, "\delta^t\Downarrow"] & {} \\ & & a \end{tikzcd}
$$

となるので，$\langle u^t, \delta^t\rangle$ の普遍性により $h \circ f = f^t$ である．また $h\varepsilon = \varepsilon^t_h$ も u^t の普遍性から分かる． □

この定理で得られる h を随伴 $f \dashv u$ の right comparison といい，right comparison が同型なとき $f \dashv u$ は monadic であるという．

定義. t をモナドとする．$\mathcal{C}^{\mathrm{op}}$ での左 t-加群を右 t-加群という．

$$
\begin{tikzcd} & a \arrow[dl, "h"'] \\ x \arrow[r, phantom, "\xi\Downarrow"] & {} \\ & a \arrow[ul, "h"] \arrow[uu, "t"'] \end{tikzcd}
$$

$\mathcal{C}^{\mathrm{op}}$ での普遍左 t-加群 $\langle f_t\colon a \to a_t,\ \varphi\colon f_t \circ t \Rightarrow f_t\rangle$ を普遍右 t-加群という．また a_t を t の Kleisli 対象という．

定理 7. $\mathrm{Alg}_{\mathcal{C}^{\mathrm{op}}}$ が存在する $\Longleftrightarrow$ $\mathcal{C}$ の任意のモナド t の Kleisli 対象が存在する． □

定理 8. $\mathrm{Alg}_{\mathcal{C}^{\mathrm{op}}}$ が存在するならば，$\mathcal{C}$ の任意のモナド $\langle a, t, \mu, \eta\rangle$ は $\mathcal{C}$ の随伴から得られる．

証明. $\mathrm{Alg}_{\mathcal{C}^{\mathrm{op}}}$ が存在するとする．t を $\mathcal{C}$ のモナドとすると，これは $\mathcal{C}^{\mathrm{op}}$ のモナドでもあるから，t は $\mathcal{C}^{\mathrm{op}}$ の随伴 $u \dashv f$ から得られる．このとき $\mathcal{C}$ の随伴 $f \dashv u$ が成り立ち，これから t が得られる． □

$\mathrm{Alg}_{\mathcal{C}^{\mathrm{op}}}$ が存在するとして $f \dashv u\colon a \longrightarrow b$ を随伴とする．$\mathcal{C}^{\mathrm{op}}$ の随伴 $u \dashv f\colon a \longrightarrow b$ から得られる $\mathcal{C}^{\mathrm{op}}$ のモナドを t とすると，t から $\mathcal{C}^{\mathrm{op}}$ の随伴 $u_t \dashv f_t\colon a \longrightarrow a_t$ が得られる．このとき $\mathcal{C}^{\mathrm{op}}$ の 1-morphism $h\colon b \longrightarrow a_t$ が存在して $f_t \circ h = f$，$h \circ u = u_t$ となる．

$$
\begin{tikzcd} & a \arrow[dl, bend right, "u"'] \arrow[dr, bend left=15, "u_t"'] & \\ b \arrow[ur, bend right=15, "f"'] \arrow[rr, dashed, "h"'] & & a_t \arrow[ul, bend right, "f_t"'] \end{tikzcd}
\quad \text{in } \mathcal{C}^{\mathrm{op}}
$$

即ち $\mathcal{C}$ において随伴 $f_t \dashv u_t \colon a \longrightarrow a_t$ が存在して，$h \circ f_t = f$，$u \circ h = u_t$ である．

$$
\begin{tikzcd}
 & a \arrow[dl, bend right, "f_t"'] \arrow[dr, bend left, "f"] & \\
a_t \arrow[ur, bend right, "u_t"'] \arrow[ur, phantom, "\perp"] \arrow[rr, dashed, "h"'] & & b \arrow[ul, bend right, "u"'] \arrow[ul, phantom, "\top"]
\end{tikzcd}
$$

定理 9. $\mathrm{Alg}_{\mathbf{Cat}}$ が存在する．故に $\mathbf{Cat}$ の任意のモナドは随伴関手から得られる．

証明. A を圏として，モナド $\langle A, T, \mu, \eta \rangle$ に対して Eilenberg-Moore 対象 A^T を構成すればよい．

※ A^T が存在したとすると $A^T \cong \mathbf{Cat}(\mathbf{1}, A^T) \cong \mathrm{Monad}(\mathbf{Cat})(\mathrm{Inc}(\mathbf{1}), T)$ となる．よって圏 A^T の対象は左 T-加群 $\langle H \colon \mathbf{1} \to A, \xi \colon TH \Rightarrow H \rangle$ である．

$$
\begin{tikzcd}
 & A \arrow[d, "T"] \\
\mathbf{1} \arrow[ur, "H"] \arrow[dr, "H"'] \arrow[r, phantom, "\xi\Downarrow"] & A
\end{tikzcd}
$$

A の対象 $a := H(*)$ と A の射 $k := \xi_* \colon TH(*) \longrightarrow H(*)$ は次の可換図式を満たす．

$$
\begin{tikzcd}
TTa \arrow[r, "Tk"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "k"] \\
Ta \arrow[r, "k"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "k"] \\
 & a
\end{tikzcd}
$$

逆に $\langle a, k \rangle$ が上記の条件を満たせば，左 T-加群を与えることが分かる．

圏 A^T を次のように定義する．（この A^T を Eilenberg-Moore 圏という．）

- $a \in A$ と射 $k \colon Ta \longrightarrow a$ の組 $\langle a, k \rangle$ で，次の図式を可換にするものを対象とする．

$$
\begin{tikzcd}
TTa \arrow[r, "Tk"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "k"] \\
Ta \arrow[r, "k"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "k"] \\
 & a
\end{tikzcd}
$$

- $\langle a,k\rangle$ から $\langle a',k'\rangle$ と射とは，射 $f\colon a\longrightarrow a'$ で次の図式を可換にするものとする．

$$\begin{tikzcd}
Ta \arrow[r, "k"] \arrow[d, "Tf"'] & a \arrow[d, "f"] \\
Ta' \arrow[r, "k'"'] & a'
\end{tikzcd}$$

関手 $U^T\colon A^T\longrightarrow A$ を $U^T(\langle a,k\rangle):=a, U^T(f):=f$ で定める．自然変換 $\delta^T\colon T\circ U^T\Longrightarrow U^T$ を $\delta^T_{\langle a,k\rangle}:=k\colon Ta\longrightarrow a$ で定める．$\langle U^T,\delta^T\rangle$ は左 T-加群である．

$$\begin{tikzcd}
 & A \arrow[d, "T"] \\
A^T \arrow[ur, "U^T"] \arrow[dr, "U^T"'] \arrow[r, phantom, "\delta^T\Downarrow"] & A
\end{tikzcd}$$

$\langle U^T,\delta^T\rangle$ が普遍左 T-加群であることを示す．$\langle H,\xi\rangle$ が左 T-加群であるとする．

$$\begin{tikzcd}
 & A \arrow[d, "T"] \\
X \arrow[ur, "H"] \arrow[dr, "H"'] \arrow[r, phantom, "\xi\Downarrow"] & A
\end{tikzcd}$$

関手 $F\colon X\longrightarrow A^T$ を

- 対象 $x\in X$ に対して $Fx:=\langle Hx,\xi_x\rangle$ とする．($\langle H,\xi\rangle$ が左 T-加群だから，次の図式が可換となる．)

$$\begin{tikzcd}
TTa \arrow[r, "Tk"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "k"] \\
Ta \arrow[r, "k"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "k"] \\
 & a
\end{tikzcd}$$

- X の射 $f\colon x\longrightarrow y$ に対して $Ff:=Hf$ とする．(ξ が自然変換だから次の図式が可換であり，よって $f\colon\langle Hx,\xi_x\rangle\longrightarrow\langle Hy,\xi_y\rangle$ は A^T の射である．)

$$\begin{tikzcd}
THx \arrow[r, "\xi_x"] \arrow[d, "THf"'] & Hx \arrow[d, "Hf"] \\
THy \arrow[r, "\xi_y"'] & Hy
\end{tikzcd}$$

で定める．このとき $U^T \circ F = H$，$\delta^T \circ F = \xi$ である．

$$
\begin{tikzcd}
 & & A \arrow[dd, "T"] \\
X \arrow[r, dashed, "F"] \arrow[urr, bend left, "H"] \arrow[drr, bend right, "H"'] & A^T \arrow[ur, "U^T"] \arrow[dr, "U^T"'] \arrow[r, phantom, "\delta^T \Downarrow"] & {} \\
 & & A
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[dd, "T"] \\
X \arrow[ur, bend left, "H"] \arrow[dr, bend right, "H"'] \arrow[r, phantom, "\xi \Downarrow"] & {} \\
 & A
\end{tikzcd}
$$

明らかにこのような F は一意である．

$\langle H, \xi \rangle$，$\langle H', \xi' \rangle$ を左 T-加群として，自然変換 σ が次の等式を満たすとする．

$$
\begin{tikzcd}
 & A \arrow[dd, "T"] \\
X \arrow[ur, bend left, "H"] \arrow[ur, bend right, "H'"'] \arrow[dr, bend right, "H'"'] \arrow[r, phantom, "\Downarrow\sigma \quad \xi'\Downarrow"] & {} \\
 & A
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[dd, "T"] \\
X \arrow[ur, bend left, "H"] \arrow[dr, bend left, "H"] \arrow[dr, bend right, "H'"'] \arrow[r, phantom, "\xi\Downarrow \quad \Downarrow\sigma"] & {} \\
 & A
\end{tikzcd}
$$

$\langle H, \xi \rangle$，$\langle H', \xi' \rangle$ に対して上記方法で $F, F' \colon X \longrightarrow A^T$ を定義する．このとき $\tau \colon F \longrightarrow F'$ を $\tau_x := \sigma_x \colon \langle Hx, \xi_x \rangle \longrightarrow \langle H'x, \xi'_x \rangle$ で定める．すると明らかに $U^T \circ \tau = \sigma$ であり，このような τ は一意である．

以上により $\langle U^T, \delta^T \rangle$ が普遍左 T-加群であって，A^T が Eilenberg-Moore 対象であることが分かった． □

定理 10. $\mathrm{Alg}_{\mathbf{Cat}^{\mathrm{op}}}$ が存在する．

証明. A を圏として，モナド $\langle A, T, \mu, \eta \rangle$ に対して Kleisli 対象 A_T を構成すればよい．圏 A_T を以下のように定義する．(この A_T を Kleisli 圏という．)

- $\mathrm{Ob}(A_T) := \mathrm{Ob}(A)$ とする．
- $a, b \in \mathrm{Ob}(A_T)$ に対して，a から b への射とは，A の射 $a \longrightarrow Tb$ のこととする．A の射 $f \colon a \longrightarrow Tb$ に対応する A_T の射を $f^\flat \colon a \longrightarrow b$ で表す．
- 合成は $(a \xrightarrow{f^\flat} b \xrightarrow{g^\flat} c) := (a \xrightarrow{f} Tb \xrightarrow{Tg} TTc \xrightarrow{\mu_c} Tc)^\flat$ で定義する．

関手 $F_T \colon A \longrightarrow A_T$ を

- 対象 $a \in A$ に対して $F_T(a) := a$ とする．
- $f \colon a \longrightarrow b$ に対して $F_T(f) := (a \xrightarrow{f} b \xrightarrow{\eta_b} Tb)^\flat$ とする．

で定めて，自然変換 $\varphi \colon F_T \circ T \Longrightarrow F_T$ を $\varphi_a := (\mathrm{id}_{Ta})^\flat \colon Ta \longrightarrow a$ により定めると

$\langle F_T, \varphi\rangle$ は右 T-加群である．

$\langle F_T, \varphi\rangle$ が普遍右 T-加群であることを示せばよい．その為に $\langle H, \xi\rangle$ を右 T-加群とする．

$$\begin{tikzcd}[row sep=small] & A \arrow[dl, "H"'] \\ X \arrow[r, phantom, "\xi\Downarrow"] & \\ & A \arrow[ul, "H"] \arrow[uu, "T"'] \end{tikzcd}$$

関手 $F\colon A_T \longrightarrow X$ を

- 対象 $a \in A_T$ に対して $F(a) := H(a)$ とする．
- $f^\flat\colon a \longrightarrow b$ に対して $F(f^\flat) := (Ha \xrightarrow{Hf} HTb \xrightarrow{\xi_b} Hb)$ とする．

で定めると次の等式が成り立つ．

$$\begin{tikzcd}[row sep=small] & & A \arrow[dl, "F_T"'] \arrow[dll, bend right, "H"'] \\ X & A_T \arrow[l, dashed, "F"'] \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[r, phantom, "\varphi\Downarrow"] & \\ & & A \arrow[ul, "F_T"] \arrow[ull, bend left, "H"] \arrow[uu, "T"'] \end{tikzcd} = \begin{tikzcd}[row sep=small] & A \arrow[dl, bend right, "H"'] \\ X \arrow[r, phantom, "\xi\Downarrow"] & \\ & A \arrow[ul, bend left, "H"] \arrow[uu, "T"'] \end{tikzcd}$$

明らかにこのような F は一意である．

$\langle H, \xi\rangle$，$\langle H', \xi'\rangle$ を右 T-加群として，自然変換 σ が次の等式を満たすとする．

$$\begin{tikzcd}[row sep=small] & A \arrow[dl, bend right, "H"'] \arrow[dl, bend left, "H'"] \arrow[dl, phantom, "\Downarrow\sigma"] \\ X \arrow[r, phantom, "\xi'\Downarrow"] & \\ & A \arrow[ul, bend left, "H'"] \arrow[uu, "T"'] \end{tikzcd} = \begin{tikzcd}[row sep=small] & A \arrow[dl, bend right, "H"'] \\ X \arrow[r, phantom, "\xi\Downarrow"] & \\ & A \arrow[ul, bend left, "H'"] \arrow[ul, bend right, "H"'] \arrow[ul, phantom, "\Downarrow\sigma"] \arrow[uu, "T"'] \end{tikzcd}$$

$\langle H, \xi\rangle$，$\langle H', \xi'\rangle$ に対して上記方法で $F, F'\colon A_T \longrightarrow X$ を定義する．このとき $\tau\colon F \longrightarrow F'$ を $\tau_a := \sigma_a\colon Fa = Ha \longrightarrow Hb = Fb$ で定める．すると明らかに $\tau \circ F_T = \sigma$ であり，このような τ は一意である．

以上により $\langle F_T, \varphi\rangle$ が普遍右 T-加群であって，A_T が Kleisli 対象であることが分かった． $\square$

定義． t をコモナドとする．$\mathcal{C}^{\mathrm{co}}$ での左 t-加群を左 t-余加群という．

$$\begin{tikzcd}[row sep=small] & a \arrow[dd, "t"] \\ x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, phantom, "\xi\Uparrow"] & \\ & a \end{tikzcd}$$

$\mathcal{C}^{\mathrm{co}}$ での普遍左 t-加群を普遍左 t-余加群という．

補題 11. $\langle a, t, \mu, \eta\rangle$ をモナド，$f\colon a \longrightarrow b$ を 1-morphism として，左 Kan 拡張 $\langle (f\circ t)^\dagger f, \eta'\rangle$ が存在するとする．

$$\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[d, "(f\circ t)^\dagger f"] \\
a \arrow[u, "t"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\eta'"] & b
\end{tikzcd}$$

このとき $(f\circ t)^\dagger f$ はコモナドになる．

証明. Kan 拡張の普遍性により次の図式の μ', ε を取る．

$$\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[dr, "(f\circ t)^\dagger f"] \arrow[dd, bend right, "(f\circ t)^\dagger f"'] & \\
 & \arrow[r, phantom, "\overset{\mu'}{\Rightarrow}"] & b \arrow[dl, "(f\circ t)^\dagger f"] \\
a \arrow[uu, bend left, "t"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\eta'"] & b &
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "f"] & & b \arrow[dr, "(f\circ t)^\dagger f"] & \\
 & a \arrow[ul, "t"'] \arrow[rr, "f"'] \arrow[urr, phantom, "\Uparrow\eta'"] & & b \arrow[dl, "(f\circ t)^\dagger f"] \\
a \arrow[uu, bend left, "t"] \arrow[ur, "t"] \arrow[rr, "f"'] \arrow[u, phantom, "\overset{\mu}{\Leftarrow}"] \arrow[urr, phantom, "\Uparrow\eta'"] & & b &
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[d, bend right, "(f\circ t)^\dagger f"'] \arrow[d, bend left, "\mathrm{id}_b"] \arrow[d, phantom, "\overset{\varepsilon}{\Rightarrow}"] \\
a \arrow[u, bend left, "t"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\eta'"] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, bend left, "\mathrm{id}_a"] & b \arrow[d, bend left, "\mathrm{id}_b"] \\
a \arrow[u, bend left, "t"] \arrow[u, phantom, "\overset{\eta}{\Leftarrow}"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\mathrm{id}_f"] & b
\end{tikzcd}$$

このとき $\langle a, (f\circ t)^\dagger f, \mu', \varepsilon\rangle$ がコモナドとなることが分かる． □

定理 12. $\mathcal{C}$ に米田構造が入っているとする．a を admissible で $t\colon a \longrightarrow a$ をモナドとする．このとき前補題により $t^{-1} = (y_a\circ t)^\dagger y_a$ はコモナドである．次の図式の $\langle v, \psi\rangle$ を普遍左 t^{-1}-余加群とする．

$$\begin{tikzcd}
 & \widehat{a} \arrow[d, "t^{-1}"] \\
c \arrow[ur, "v"] \arrow[r, phantom, "\psi\Uparrow"] \arrow[dr, "v"'] & {} \\
 & \widehat{a}
\end{tikzcd}$$

このとき $e \in \mathcal{C}$ が t の Eilenberg-Moore 対象 $\Longleftrightarrow$ 次の形の pullback が存在する．

$$\begin{tikzcd}
e \arrow[r] \arrow[d] & c \arrow[d, "v"] \\
a \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}$$

証明. ($\Longrightarrow$) $\langle u^t \colon a^t \to a,\ \varphi\rangle$ を普遍左 t-加群とする．$a(t,1) \cong t^{-1} \circ a(1,1) = t^{-1} \circ y_a$ だった．次の 2-morphism の合成を θ とする．

$$\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[dr, "a(t{,}1)" description] \arrow[dr, phantom, shift left=3, "\cong"] & \widehat{a} \arrow[d, "t^{-1}"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\varphi\Uparrow"] & \arrow[dr, phantom, "\Uparrow\chi^t"] & \\
 & a \arrow[uu, "t"'] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}$$

$\langle y_a \circ u^t, \theta\rangle$ は t^{-1}-余代数である．よって普遍 t^{-1}-余代数 $\langle v, \psi\rangle$ の普遍性から次の $f \colon a^t \longrightarrow c$ が存在する．

$$\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, dashed, "f"] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[dr, "a(t{,}1)" description] \arrow[dr, phantom, shift left=3, "\cong"] & \widehat{a} \arrow[d, "t^{-1}"] \\
a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\varphi\Uparrow"] & \arrow[dr, phantom, "\Uparrow\chi^t"] & \\
 & a \arrow[uu, "t"'] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}$$

このとき左辺の左下の四角が pullback であることを示せばよい．そこで次の図式の x を取る．

$$\begin{tikzcd}
x \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_0"'] & & \\
 & a^t \arrow[r, "f"'] \arrow[d, "u^t"] & c \arrow[d, "v"] \\
 & a \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}$$

このとき次の図式を得るが，これは t^{-1}-余代数を与える．

$$\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[ur, "q_0"] \arrow[dr, "q_0"'] \arrow[r, "q_1"] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}$$

絶対左 Kan リフト $a(t,1)_\dagger y_a = \langle t, \chi^t \rangle$ の普遍性から次の ξ を得る.

$$
\begin{tikzcd}
 & a \arrow[ddr, bend left=90, "{a(t,1)}"] & \\
x \arrow[ur, "q_0"] \arrow[dr, "q_0"'] \arrow[r, phantom, "\xi\Uparrow"] & \arrow[r, phantom, "\chi^t\Uparrow"] & {} \\
 & a \arrow[uu, "t"'] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] \arrow[ddr, bend left=90, "{a(t,1)}"] & \widehat{a} \arrow[dd, "t^{-1}"] \arrow[dd, phantom, "\cong", shift left=6] \\
x \arrow[ur, "q_0"] \arrow[r, "q_1"] \arrow[dr, "q_0"'] & c \arrow[ur, "v"'] \arrow[dr, "v"] \arrow[r, phantom, "\psi\Uparrow"] & {} \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

すると $\langle q_0, \xi \rangle$ は左 t-加群である. よって a^t の普遍性から $r\colon x \longrightarrow a^t$ が一意に存在して次の等式が成り立つ.

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[urr, bend left, "q_0"] \arrow[drr, bend right, "q_0"'] \arrow[r, dashed, "r"] & a^t \arrow[ur, "u^t"] \arrow[dr, "u^t"'] \arrow[r, phantom, "\varphi\Downarrow"] & {} \\
 & & a
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "q_0"] \arrow[dr, bend right, "q_0"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

故に $u^t \circ r = q_0$ であるから, $f \circ r = q_1$ を示せばよい.

$$
\begin{tikzcd}
x \arrow[dr, dashed, "r"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_0"'] & & \\
 & a^t \arrow[r, "f"'] \arrow[d, "u^t"] & c \arrow[d, "v"] \\
 & a \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

それは定義から $v \circ f \circ r = y_a \circ u^t \circ r = y_a \circ q_0 = v \circ q_1$ となるので, v の普遍性から $f \circ r = q_1$ が分かる.

($\Longleftarrow$) 次の図式を pullback とする.

$$
\begin{tikzcd}
e \arrow[r, "f"] \arrow[d, "u"'] & c \arrow[d, "v"] \\
a \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

次の合成は t^{-1}-余代数である．

$$
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
e \arrow[ur, "u"] \arrow[r, "f"] \arrow[dr, "u"'] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & {} \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

$t_{\dagger}^{-1}y_a = \langle y_a \circ t, \chi^{y_a \circ t}\rangle$ が絶対左 Kan リフトだから次の等式を満たす θ を得る．

$$
\begin{tikzcd}
 & a \arrow[rr, "y_a"] \arrow[dd, phantom, "\Uparrow\theta"] & & \widehat{a} \arrow[dd, "t^{-1}"] \\
e \arrow[ur, "u"] \arrow[dr, "u"'] & & a \arrow[ur, "y_a"] & \\
 & a \arrow[ur, "t"] \arrow[rr, "y_a"'] \arrow[urr, phantom, "\chi^{y_a \circ t}\Uparrow"] & & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
e \arrow[ur, "u"] \arrow[r, "f"] \arrow[dr, "u"'] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & {} \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

$y_{a\dagger}y_a = \langle \mathrm{id}_a, \mathrm{id}_{y_a}\rangle$ が絶対左 Kan リフトだから，次の等式が成り立つ φ が取れる．

$$
\begin{tikzcd}
 & a \arrow[rr, "y_a"] \arrow[dd, phantom, "\Uparrow\varphi"] & & \widehat{a} \\
e \arrow[ur, "u"] \arrow[dr, "u"'] & & a \arrow[ul, "\mathrm{id}_a"'] \arrow[ur, "y_a"'] \arrow[u, phantom, "\Uparrow\mathrm{id}"] & \\
 & a \arrow[ur, "t"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[rr, "y_a"] \arrow[dd, phantom, "\Uparrow\theta"] & & \widehat{a} \\
e \arrow[ur, "u"] \arrow[dr, "u"'] & & a \arrow[ur, "y_a"'] & \\
 & a \arrow[ur, "t"'] & &
\end{tikzcd}
$$

この $\langle u, \varphi\rangle$ は左 t-加群である．

$\langle u, \varphi\rangle$ が普遍左 t-加群であることを示すため，$\langle h\colon x \to a, \xi\rangle$ を左 t-加群とする．$\Longrightarrow$ の証明で $\langle u^t, \varphi\rangle$ から f を得たのと同じ方法で，次の f' を得る．

$$
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[ur, "h"] \arrow[r, dashed, "f'"] \arrow[dr, "h"'] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & {} \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
=
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[ddr, "a(t,1)"] \arrow[r, phantom, shift right=3, "\cong"] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, phantom, "\xi\Uparrow"] & {} \arrow[r, phantom, "\Uparrow\chi^t"] & {} \\
 & a \arrow[uu, "t"] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

左辺の左下の四角が可換で，e が pullback だったから射 $r\colon x \longrightarrow e$ が一意に存在して次の図式が可換となる．

$$
\begin{tikzcd}
x \arrow[dr, dashed, "r"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "h"'] & & \\
 & e \arrow[r, "f"] \arrow[d, "u"] & c \arrow[d, "v"] \\
 & a \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

このとき，次の等式が成り立つことを示せばよい．

$$
\begin{tikzcd}
 & & a \arrow[dd, "t"] \\
x \arrow[r, dashed, "r"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & e \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, phantom, "\varphi\Downarrow"] & {} \\
 & & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & a \arrow[dd, "t"] \\
x \arrow[ur, bend left, "h"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\xi\Downarrow"] & {} \\
 & a
\end{tikzcd}
$$

それは

$$
\begin{tikzcd}
 & & a \arrow[r, "y_a"] \arrow[ddr, "\cong", "a(t,1)"'] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[r, "r"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & e \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, phantom, "\varphi\Uparrow"] & {} \arrow[r, phantom, "\Uparrow\chi^t"] & {} \\
 & & a \arrow[uu, "t"] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & a \arrow[rr, "y_a"] & & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[r, "r"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & e \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, phantom, "\varphi\Uparrow"] & {} & a \arrow[ul, "\mathrm{id}_a"'] \arrow[ur, "y_a"'] \arrow[u, phantom, "\Uparrow\mathrm{id}"] \arrow[d, phantom, "\chi^{y_a\circ t}\Uparrow"] & {} \\
 & & a \arrow[ur, "t"] \arrow[rr, "y_a"'] & & \widehat{a}
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[r, "r"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & e \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, "f"] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & {} \\
 & & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[d, equal] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, "f'"] & c \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[r, phantom, "\psi\Uparrow"] & {} \\
 & a \arrow[u, equal] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & a \arrow[r, "y_a"] \arrow[ddr, "\cong", "a(t,1)"'] & \widehat{a} \arrow[dd, "t^{-1}"] \\
x \arrow[ur, "h"] \arrow[dr, "h"'] \arrow[r, phantom, "\xi\Uparrow"] & {} \arrow[r, phantom, "\Uparrow\chi^t"] & {} \\
 & a \arrow[uu, "t"] \arrow[r, "y_a"'] & \widehat{a}
\end{tikzcd}
$$

となるから，絶対左 Kan リフト $a(t,1)_\dagger y_a = \langle t, \chi^t \rangle$ の普遍性から分かる． $\square$

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